XMUT315 Control Systems Engineering

Note 8: Steady-State Analysis

Topic

- Steady-state error.
- Derivation of steady-state error.
- Steady-state analysis of step input.
- Steady-state analysis of ramp input.
- Steady-state analysis of parabola input.
- Steady-state error of other types of system.
- Sensitivity of system parameters towards steady-state errors.

1. Steady State Analysis

Steady-state analysis looking at the condition and characteristics of the system as it is settling down. It is a specialised type of time domain analysis with regard analysis of the steady-state condition of the systems.

1.1. Error in Control Systems

For the following feedback control system, the system error $e(t)$ for a feedback control system is given by the difference between the demanded output $r(t)$ and the actual output $c(t)$.

$$e(t) = r(t) - c(t)$$

The error in the equation above is illustrated in the following diagram.

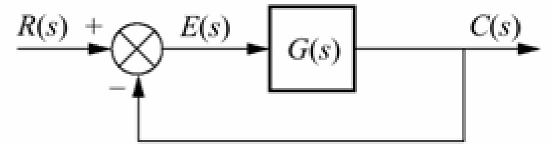


Figure 1: Error in feedback control system

1.2. Steady-State Error

The steady-state error is then defined as the difference between demanded and actual output when the system is in steady-state e.g. $t \rightarrow \infty$.

In this course, the steady-state error is now defined for specific test inputs (there are other types of input tests available in control system engineering e.g. sinusoidal, square wave, etc.):

- Step input
- Ramp input
- Parabola input

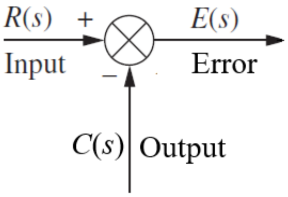
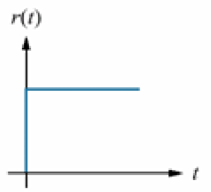
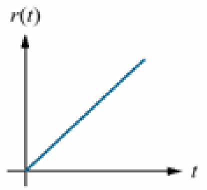


Figure 2: Steady-state errors

1.3. Test Inputs for Steady-State Error Analysis

For steady-state analysis, the following table lists and described common test inputs: step, ramp and parabola.

Name	Waveform	Physical Interpretation	Time Function	Laplace Transform
Step		Constant position	1	$\frac{1}{s}$
Ramp		Constant velocity	t	$\frac{1}{s^2}$

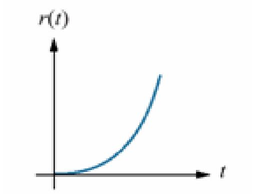
Parabola		Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

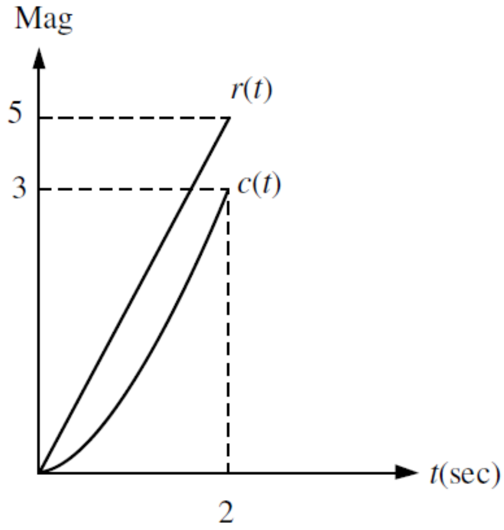
Figure 3: Test inputs for steady-state error analysis

Example for Tutorial 1 – Steady-State Errors

Figure given below shows the ramp input $r(t)$ and the output $c(t)$ of a system. Assuming the output's steady state can be approximated by a ramp, find:

a. Steady-state error. [2 marks]

b. Steady-state error if the input becomes $r(t) = tu(t)$. [2 marks]



Answer

a. From the figure, the steady-state error of the system is:

$$e(\infty) = r(\infty) - c(\infty) = 5 - 3 = 2$$

b. Since the system is linear, and because the original input was $r(t) = 2.5tu(t)$, the new steady-state error is:

$$e(\infty) = \frac{2}{2.5} = 0.8$$

2. Steady-State Errors

Depending on the input reference, steady-state errors refer to the difference between the output of the system with the intended final value outcome.

2.1. Steady-State Error of Step Input

For a step input, compare the time response of different systems:

- Output 1: $e_1(\infty) = 0$ because Output 1 is equal to Input at $t = \infty$ and the steady-state error is thus zero.

- Output 2: $e_2(\infty) \neq 0$ because Output 2 is NOT equal to Input at $t = \infty$ and the steady-state error is thus non-zero.

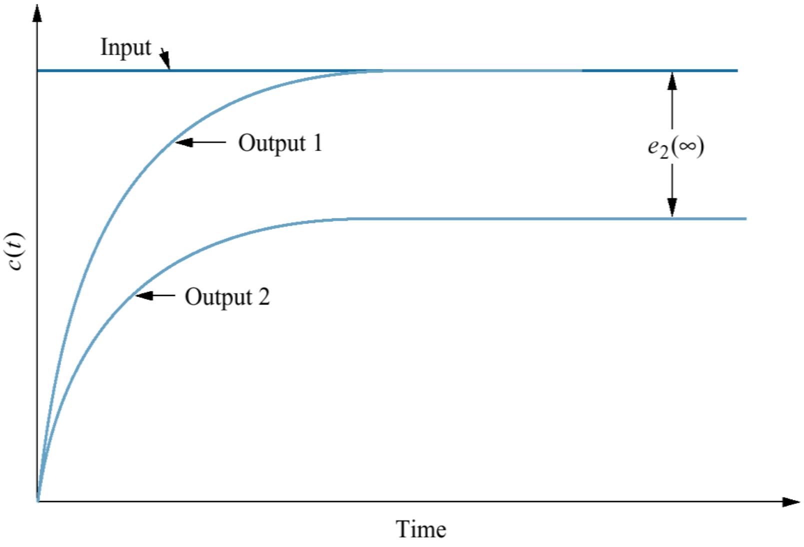


Figure 4: Steady state error of step input

2.2. Steady-State Error of Ramp Input

For a ramp input:

- Output 1: $e_1(\infty) = 0$ because Output 1 = Input at $t = \infty$ and the steady-state error is thus zero.

- Output 2: Although the response has the same slope as the ramp input, $e_2(\infty) \neq 0$ because there will be a finite error at $t = \infty$ and the steady-state error is thus non-zero.

- Output 3: $e_3(\infty) = \infty$ because the error will increase with time as the response has a different slope than the ramp input.

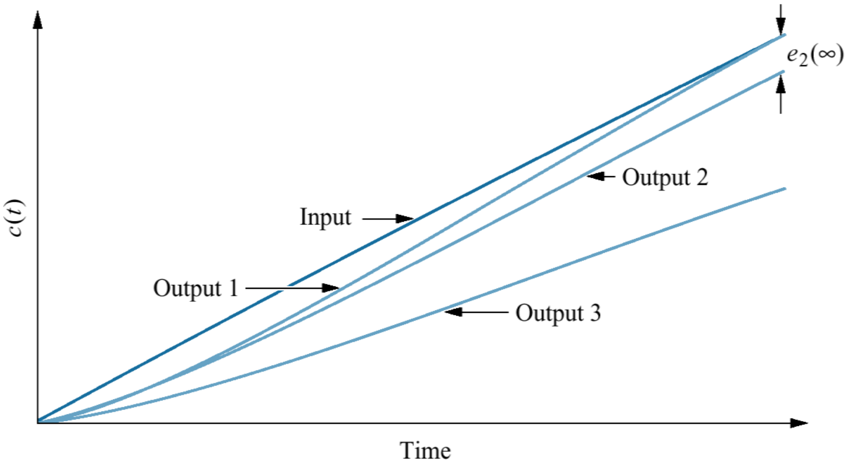


Figure 5: Steady-state error of ramp input

2.3. General Closed Loop (Unity Feedback System)

The system error (in both cases) is then given as by the definition as:

$$E(s) = R(s) - C(s)$$

We will now derive expressions for the steady-state error in unit feedback systems and then expand to non-unity feedback.

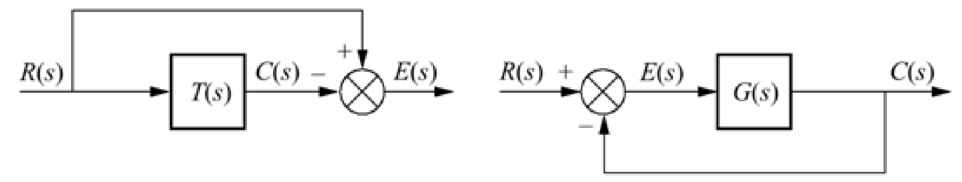


Figure 6: General closed loop systems

2.4. Sources of Steady-State Error

Consider steady-state errors due to system configuration. System with pure gain element.

System output:

$$C(s) = KE(s)$$

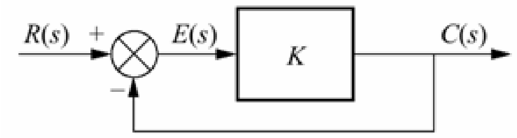


Figure 7: Feedback control system

The steady-state error can then never be equal to zero, nor the output of the system will be zero.

There will thus always be a steady-state error present. If C_{ss} is the steady-state value of the output and e_{ss} is the steady-state value of the error, then:

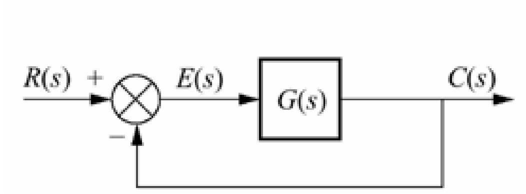
$$c_{ss}(t) = Ke_{ss}(t)$$

For a unity-feedback control system as shown above, error will diminish K increases.

Example for Tutorial 2 – Steady-State Error of Unity Feedback Systems 1

Determine the steady-state error of the unity feedback system as shown below if the plant $G(s)$ is given a step input $(1/s)$: [4 marks]

$$G(s) = \frac{2}{s(s+2)}$$



Answer

The steady-state error of the unity feedback system is determined from:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Entering the transfer-function equation of the plant to the equation above, it becomes:

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+\left[\frac{2}{s(s+2)}\right]} = \lim_{s\to 0} \frac{s(s+2)}{s(s+2)+2} = 0$$

The steady-state error of the given unity feedback system is 0.

Example for Tutorial 3 – Steady-State Error of Unity Feedback Systems 2

A Type 3 unity feedback system has applied $r(t) = 10t^3$ to its input.

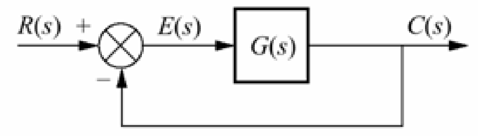
a. Derive the equation for error of a unity feedback system. [6 marks]

b. Find the steady-state error for this input if the forward transfer function is: [4 marks]

$$G(s) = \frac{1030(s^2+8s+23)(s^2+21s+18)}{s^3(s+6)(s+13)}$$

Answer

a. For a given unity feedback system as given below, derive the equation for error in the system.



The error of the system is determined from:

$$E(s) = R(s) - C(s) \qquad (1)$$

And

$$C(s) = E(s)G(s) \qquad (2)$$

Thus, substituting equation (2) into (1), the equation for error in the system is:

$$E(s) = \frac{R(s)}{1 + G(s)}$$

b. Thus, steady-state error of the closed -loop system is:

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left[\frac{R(s)}{1+G(s)}\right] \\ &= \lim_{s\to 0} s\left[\frac{\left(\frac{60}{s^4}\right)}{1+\frac{1030(s^2+8s+23)(s^2+21s+18)}{s^3(s+6)(s+13)}}\right] \\ &= \lim_{s\to 0}\frac{60(s+6)(s+13)}{s^3(s+6)(s+13)+1030(s^2+8s+23)(s^2+21s+18)} \\ &= \frac{60(6)(13)}{1030(23)(18)} = 0.011 \end{aligned}$$

3. Steady-State Equation Derivation

In this section, several steady-state equations for control systems are derived.

3.1. Steady-State Error in Terms of $G(s)$

For the system:

$$E(s) = R(s) - C(s)$$

Thus:

$$E(s) = R(s) - E(s)G(s)$$

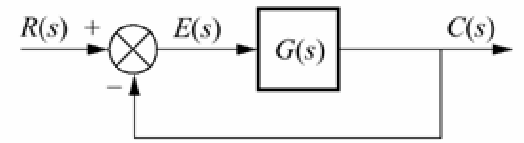


Figure 8: Steady-state error in terms of $G(s)$

So that:

$$E(s) = \frac{R(s)}{1 + G(s)}$$

From the final value theorem:

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Above equation will thus allow us to calculate the steady-state error given a particular input $R(s)$

3.2. Steady-State Error of Step Input

For a step Input, with $R(s) = 1/s$, we have:

$$e(\infty) = e_{step}(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s}\right)}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

For zero steady-state error, we need:

$$\lim_{s \to 0} G(s) = \infty$$

To satisfy the above equation, $G(s)$ must have the form:

$$G(s) = \frac{(s + z_1)(s + z_2) \dots}{s^n (s + p_1)(s + p_2) \dots}$$

The $G(s) \to \infty$ in the limit $s \to 0$, as the denominator will become zero. To have a zero steady-state error, we must have at least one pole at the origin so that $n \geq 1$.

The term s in the denominator of the equation for $G(s)$ represents an integrating element in the feedforward path. Division by s in the frequency domain represents integration in the time domain.

At least one integrating element must be present in the forward path in order to ensure a zero steady-state error. If there are no integrations, then $n = 0$ and

$$\lim_{s \to 0} G(s) = \frac{z_1 z_2 \dots}{p_1 p_2 \dots}$$

This will be finite and will thus produce a finite steady-state error.

In order to have a zero steady-state error for a step input, we thus need at least one integrating element in the forward path.

3.3. Steady-State Error of Ramp Input

For a ramp input, we have $r(t) = tu(t)$, where $r(t) = t$ for $t > 0$ and $r(t) = 0$ elsewhere. With $R(s) = 1/s^2$ we have:

$$e(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s^2}\right)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s \to 0} sG(s)}$$

In order to have zero steady-state error, we need:

$$\lim_{s \to 0} sG_0(s) = \infty$$

For this condition, we need $n \geq 2$, i.e. we need at least two integrators in the open-loop transfer function.

If there is one integrator ($n = 1$):

$$\lim_{s \to 0} sG_0(s) = \frac{sKz_1z_2 \dots}{sp_1p_2 \dots} = \text{finite}$$

This will lead to a finite steady-state error.

If there are no integrators ($n = 0$):

$$\lim_{s \to 0} sG_0(s) = \frac{sKz_1z_2 \dots}{p_1p_2 \dots} = 0$$

So that, we have an infinite steady-state error.

3.4. Steady-State Error of Parabolic Input

For a parabolic input, we have:

$$r(t) = 0.5t^2$$

Thus, $R(s) = 1/s^3$, the steady-state error is then:

$$e(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s^3}\right)}{1 + G(s)} = \lim_{s \to 0} \frac{1}{s^2 + s^2G(s)} = \frac{1}{\lim_{s \to 0} s^2G(s)}$$

In order to have zero steady-state error, we need:

$$\lim_{s \to 0} s^2G_0(s) = \infty$$

We will thus require three integrators in the open-loop transfer function $n \geq 3$. If $n = 2$, there will be a finite steady-state error and for $n < 2$ there will be an infinite steady-state error.

3.5. Summary of Steady-State Errors

Expressions for the steady-state error (for unity feedback) to different inputs:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

Where:

$$e_{step}(\infty) = \frac{1}{1 + \lim\limits_{s\to 0} G(s)}$$

$$e_{ramp}(\infty) = \frac{1}{1 + \lim\limits_{s\to 0} sG(s)}$$

$$e_{parabola}(\infty) = \frac{1}{1 + \lim\limits_{s\to 0} s^2G(s)}$$

For a zero steady-state error, we need:

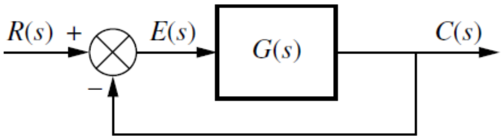
- At least one integrator in the transfer function for a step.
- At least two integrators in the transfer function for a ramp.
- At least three integrators in the transfer function for a parabola.

Example for Tutorial 4 – Steady-State Errors and Inputs

For the unity feedback system shown in the figure below, where:

$$G(s) = \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}$$

Find the steady-state errors for the following test inputs: $25u(t)$, $37tu(t)$, and $47t^2u(t)$. [6 marks]



Answer

The steady-state error of the system is calculated from:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

Where:

$$G(s) = \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}$$

For step input, $25u(t)$, $R(s) = 25/s$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0}\frac{s\left(\frac{25}{s}\right)}{1+\frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

$$= \lim_{s\to 0}\frac{25[s(s+38)(s^2+2s+28)]}{s(s+38)(s^2+2s+28)+450(s+8)(s+12)(s+15)} = 0$$

For ramp input, $37tu(t)$, $R(s) = 37/s^2$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0}\frac{s\left(\frac{37}{s^2}\right)}{1+\frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

$$= \lim_{s\to 0}\frac{37}{s+\frac{450(s+8)(s+12)(s+15)}{(s+38)(s^2+2s+28)}}$$

$$= \frac{37}{\frac{450(8)(12)(15)}{(38)(28)}} = 6.075\times 10^{-2}$$

For parabolic input, $47t^2u(t)$, $R(s) = 47/s^3$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0}\frac{s\left(\frac{47}{s^3}\right)}{1+\frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

$$= \lim_{s\to 0}\frac{47}{s^2+s\left[\frac{450(s+8)(s+12)(s+15)}{(s+38)(s^2+2s+28)}\right]} = \infty$$

4. Steady-State Error Constant and Systems Type

The characteristics and behaviour of control system at steady state can be analysed through the steady-state error constant and system types.

4.1. Static Error Constants

Static error constant and system type. The term in the denominator of the definition of the steady-state error for each input type is taken to limit the steady-state error. These are then called the static error constants and are defined as follows:

- Position constant (K_p):

$$K_p = \lim_{s \to 0} G(s)$$

- Velocity constant (K_v):

$$K_v = \lim_{s \to 0} sG(s)$$

- Acceleration constant (K_a):

$$K_a = \lim_{s \to 0} s^2 G(s)$$

These constants depend on the form of $G(s)$ and will determine the value of the steady-state error. Error decreases as the value of the static error constant increases.

4.1.1. Static position error constant (K_p)

It is associated with step input signal applied to a closed-loop system. For a given step input signal:

$$R(s) = A/s \qquad \text{(Eq. 1)}$$

Steady-state error is given as:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad \text{(Eq. 2)}$$

Put equations (1) into (2):

$$e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{A}{s}\right)}{1 + G(s)H(s)} = \frac{A}{1 + \lim_{s \to 0} G(s)H(s)} = \frac{A}{1 + K_p}$$

Where: $K_p = \lim_{s \to 0} G(s)H(s)$

4.1.2. Static velocity error constant (K_v)

It is associated with ramp input signal applied to a closed loop system. The ramp input signal is:

$$R(s) = A/s^2 \qquad \text{(Eq. 3)}$$

Steady state error is given as:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad \text{(Eq. 4)}$$

Put equations (3) into (4):

$$e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{A}{s^2}\right)}{1 + G(s)H(s)} = \frac{A}{\lim\limits_{s \to 0}(1)s + \lim\limits_{s \to 0} sG(s)H(s)} = \frac{A}{K_v}$$

Where:

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

4.1.3. Static acceleration error constant (K_a)

It is associated with parabolic input signal applied to a closed loop system. The parabolic input signal is:

$$R(s) = A/s^3 \qquad \text{(Eq. 5)}$$

Steady state error is given as:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad \text{(Eq. 6)}$$

Put equations (5) into (6):

$$e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{A}{s^3}\right)}{1 + G(s)H(s)} = \frac{A}{\lim\limits_{s \to 0}(1)s^2 + \lim\limits_{s \to 0} s^2G(s)H(s)} = \frac{A}{K_a}$$

Where:

$$K_a = \lim_{s \to 0} s^2G(s)H(s)$$

Example for Tutorial 5 – Steady-State Error Constants

For a system that has the open-loop transfer function as given below.

$$G(s) = \frac{20(s+1)}{s(s+2)(s+5)}$$

a. Determine the position, velocity and acceleration error constants (K_p, K_v, and K_a) and steady-state errors. [12 marks]

b. Comment on influence of the input on the tracking of the output of the system. [2 marks]

Answer

a. The steady-state error constants and steady-state errors for the given system are calculated and determined as follows:

- Step input:

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{20(s+1)}{s(s+2)(s+5)} = \frac{20(1)}{(0)(2)(5)} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

- Ramp input:

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{(s)20(s+1)}{s(s+2)(s+5)} = \frac{(20)(1)}{(2)(5)} = 2$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{2} = 0.5$$

- Parabolic input:

$$K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{(s^2)20(s+1)}{s(s+2)(s+5)} = \frac{(0)(20)(1)}{(2)(5)} = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

b. Since the open-loop transfer function of this system has one integrator, the output of the closed-loop system can perfectly track only the unit step.

4.2. System Type

The system type is taken to be the number of integrations in the feed-forward path. The value of n in s^n of the denominator. This value of n (the system type) then determines the steady-state error of a unit feedback system for a particular type of input.

In general, the system transfer function can be written as:

$$G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^n\prod_{k=1}^{Q}(s+p_k)}$$

Where: $\prod$ denotes a multiplication of factors.

The index 'n' denotes the system type number (if n = 0, the system type is 0; if n = 1, the system type is 1, etc.)

4.3. Steady-State Error Constant & System Type

The relationships between types of inputs, steady-state error constants and system types can be summarised as in the following table:

Input	**Steady-state error formula**	**Type 0**		**Type 1**		**Type 2**	
		Static error constant	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $1/2t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Table 1: Steady-state error and constants in steady-state analysis

Example for Tutorial 6 – Steady-State Errors and System Modes

Consider the second-order system whose open-loop transfer function is given below.

$$G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

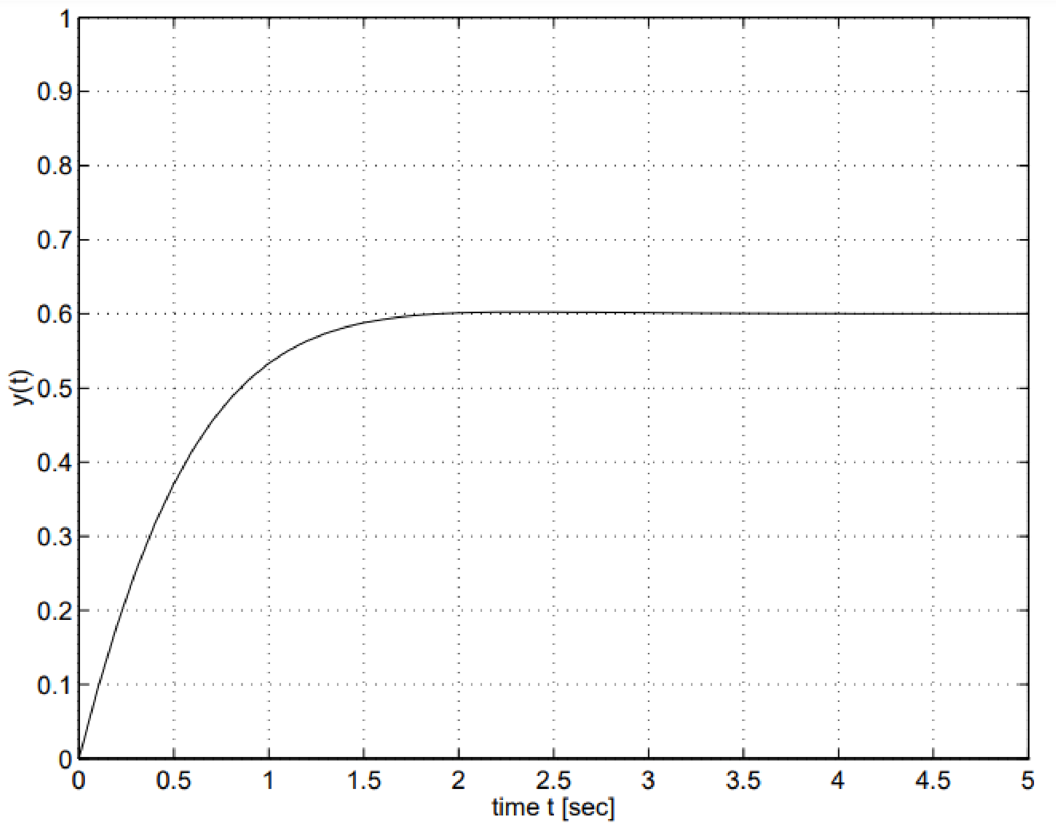
a. Sketch the time response of the system. [5 marks]

b. Calculate the position error constant (K_p) and steady-state error of the system toward unit-step input. [6 marks]

c. What type of system is the system? Can you eliminate the steady-state error of this system? [4 marks]

Answer

a. The unit-step response of the system is presented in the figure given below, from which it can be clearly seen that the steady-state output is equal to 0.6, hence the steady-state error is equal to: $e(\infty)$ = 1 - 0.6 = 0.4



b. The position error constant for this system is calculated from:

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{(s+3)}{(s+1)(s+2)} = \frac{(3)}{(1)(2)} = 1.5$$

So, that the corresponding steady-state error.

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.5} = 0.4$$

The unit-step response of the system is presented in the figure in part (a), from which it can be clearly seen that the steady-state output is equal to 0.6.

Hence, the steady-state error is equal to:

$$e(\infty) = 1 - 0.6 = 0.4$$

c. The system is a Type 0 system as it does not have any integral. The steady-state error of the system can be eliminated by introducing an integral into the system.

5. Steady-State Error for Other Types of Feedback System

We look into steady-state error for other types of feedback control system such as feedback control system with disturbances and feedback control system with non-unity feedback system.

5.1. Steady-State Error for Disturbances

Feedback control systems are often used to compensate for disturbances or unwanted inputs that enter a system.

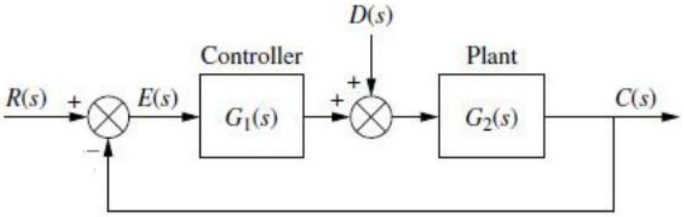


Figure 9: Feedback control system with reference input and disturbance

For a feedback control system with a disturbance, $D(s)$, injected between the controller and the plant, the transform of the output is:

$$C(s) = R(s) - E(s)$$

Thus

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

The equation for deriving steady-state error is:

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)}D(s) \quad \text{(Eq. 7)}$$

The first part is relating $E(s)$ to $R(s)$ and the second term relating $E(s)$ to $D(s)$. Apply final value theorem to find steady-state value of the error:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)}R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

The equation for the steady-state error for disturbance is:

$$e(\infty) = e_R(\infty) + e_D(\infty)$$

Where:

$$e_R(\infty) = \lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)}R(s)$$

And

$$e_D(\infty) = \lim_{s\to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)}D(s)$$

The first term $e_R(\infty)$ is the steady-state error due to $R(s)$ and the second term $e_D(\infty)$ is the steady-state error due to disturbance $D(s)$.

Assume a step disturbance $D(s) = 1/s$. Substitute this value of step disturbance into the second term of equation (7), $e_D(\infty)$, the steady-state error due to a step disturbance is:

$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\frac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

The steady-state error produced by a step disturbance can be reduced by increasing the dc gain of $G_1(s)$ or decreasing the dc gain of $G_2(s)$.

If we want to minimize the steady-state value of $E(s)$, (the output), we must increase the dc gain of $G_1(s)$ so that a lower $E(s)$ be fed back to match the steady-state value of $D(s)$, or decrease the dc value of $G_2(s)$, which yields a smaller value of $e(\infty)$, as predicted by the feedback formula.

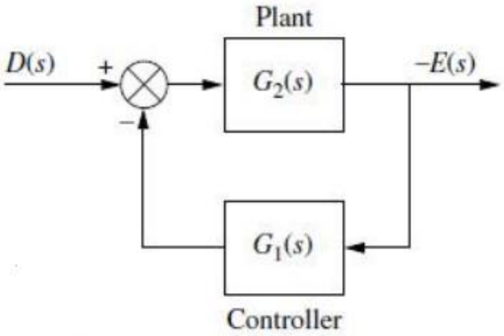
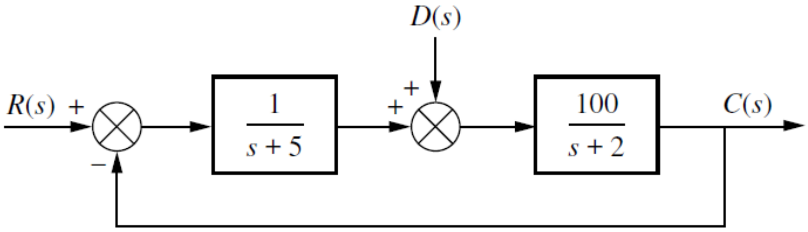


Figure 10: Rearranged feedback control system with disturbance e.g. disturbance as input and error as output (with $R(s) = 0$)

Example for Tutorial 7 – Steady-State Error for Disturbances

Find the total steady-state error due to a unit step input and a unit step disturbance in the system of the figure below. [8 marks]



Answer

From the given block diagram of the system, the equation for the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0}\frac{sR(s) - sD(s)G_2(s)}{1 + G_1(s)G_2(s)}$$

Where:

$$G_1(s) = \frac{1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{100}{s+2}$$

From the problem statement, the input signal is:

$$R(s) = D(s) = \frac{1}{s}$$

Hence, the steady-state error of the system is:

$$e(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s}\right) - s\left(\frac{1}{s}\right)\ \left(\frac{100}{s+2}\right)}{1 + \left(\frac{1}{s+5}\right)\left(\frac{100}{s+2}\right)} = \lim_{s \to 0} \frac{1 - \left(\frac{100}{s+2}\right)}{1 + \left(\frac{1}{s+5}\right)\left(\frac{100}{s+2}\right)} = -\frac{49}{11}$$

5.2. Steady-State Error for Non-Unity Feedback

A general feedback system, showing the input transducer, $G_1(s)$, controller and plant, $G_2(s)$, and feedback, $H_1(s)$, is shown in Figure (a).

Pushing the input transducer to the right past the summing junction yields the general non-unity feedback system shown in Figure (b), where $G(s) = G_1(s)G_2(s)$ and $H(s) = H_1(s)/G_1(s)$.

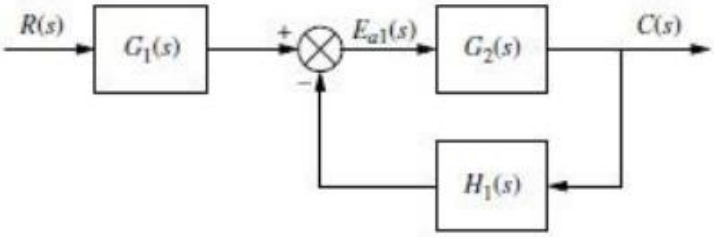
Unlike a unity feedback system, where $H(s) = 1$, the error in non-unity feedback is not the difference between the input and the output. For this case we call the signal at the output of the summing junction the actuating signal, $E_a(s)$. If $r(t)$ and $c(t)$ have the same units, we can find the steady-state error, $e(\infty) = r(\infty) - c(\infty)$.

To find out the steady-state value of the actuating signal, $E_{a1}(s)$, in Figure (a), there is no restriction that the input and output units be the same, since we are finding the steady-state difference between signals at the summing junction, which do have the same units.

The steady-state actuating signal for Figure (a) is:

$$e_{a1}(\infty) = \lim_{s \to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

The first step is to show explicitly $E(s) = R(s) - C(s)$ on the block diagram. Then, we form an equivalent unity feedback system from a general non-unity feedback system as illustrated below.



(a)

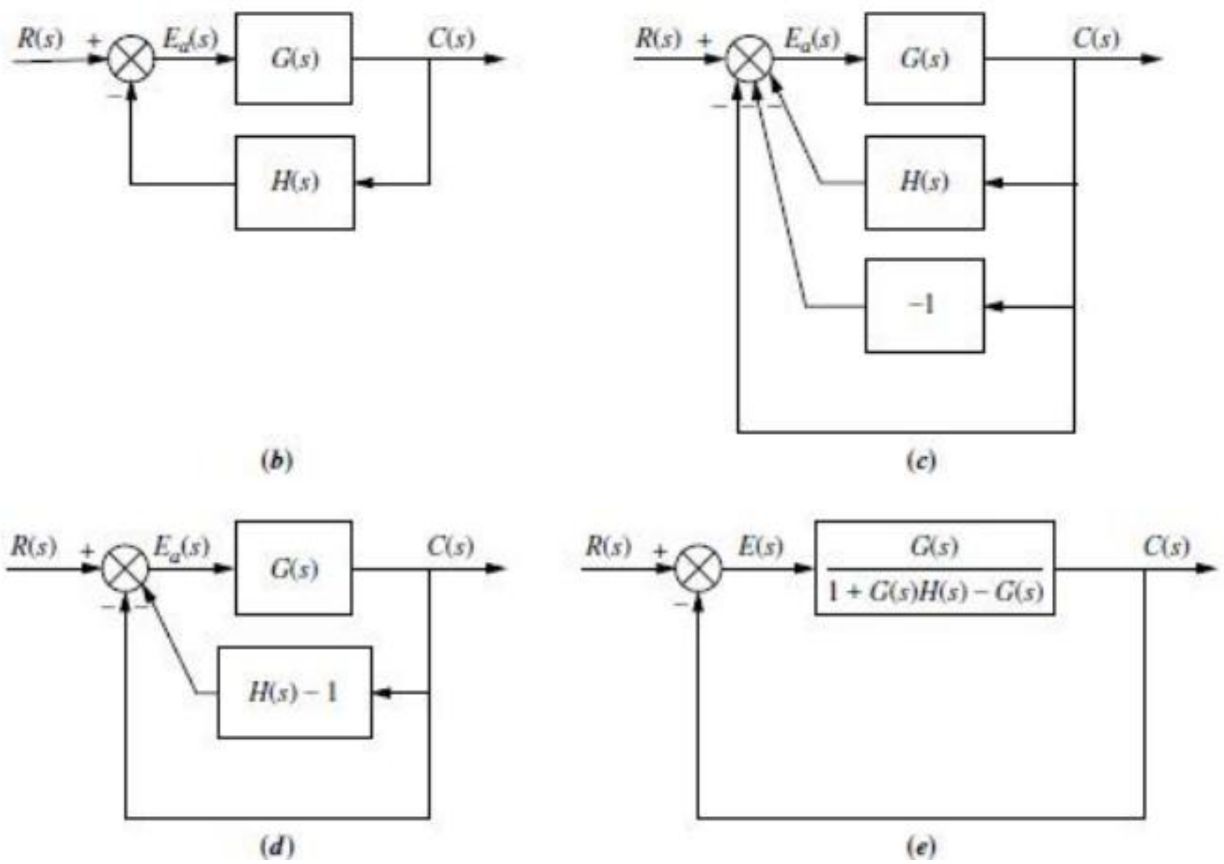


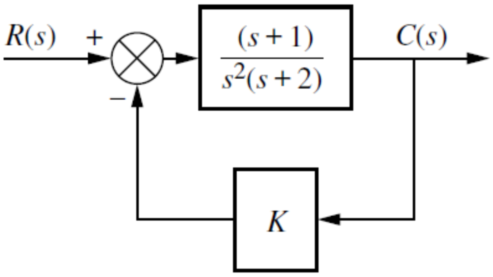
Figure 11: Forming an equivalent unity feedback system from a general non-unity feedback system

Take the non-unity feedback control system shown in Figure (b) and form a unity feedback system by adding and subtracting unity feedback paths, as shown in Figure (c). This step requires that input and output units be the same.

Next, combine $H(s)$ with the negative unity feedback, as shown in Figure (d). Finally, combine the feedback system consisting of $G(s)$ and $[H(s) - 1]$, leaving an equivalent forward path and a unity-feedback, as shown in Figure (e). Notice that the final figure shows $E(s) = R(s) - C(s)$ explicitly.

Example for Tutorial 8 – Non-Unity Feedback

Given the non-unity feedback system as shown in the figure given below, find the following:

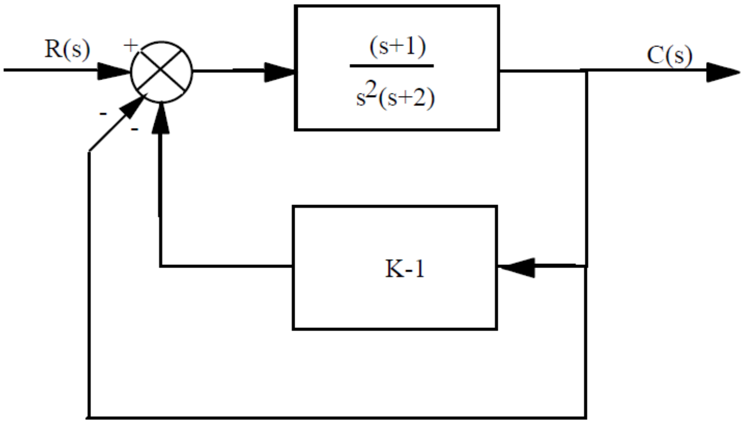


a. The system type. [4 marks]

b. The value of K to yield 0.1% error in the steady state. [14 marks]

Answer

a. Produce a unity-feedback system of the system as shown in the figure below.



Thus, the unity-feedback system of the system is:

$$G_e(s) = \frac{\frac{(s+1)}{s^2(s+2)}}{1 + \frac{(s+1)(K-1)}{s^2(s+2)}} = \frac{s+1}{s^3 + 2s^2 + (K-1)s + (K-1)}$$

As shown above, the system is Type 0.

b. Since the system is Type 0, the appropriate static error constant is K_p. Thus, the steady-state error due to step input is:

$$e_{step}(\infty) = 0.001 = \frac{1}{1 + K_p}$$

Therefore,

$$K_p = 999 = \frac{1}{K-1}$$

Hence, K = 1.001001.

Check stability: Using original block diagram, the closed-loop transfer function of the system is:

$$T(s) = \frac{\dfrac{(s+1)}{s^2(s+2)}}{1+\dfrac{K(s+1)}{s^2(s+2)}} = \frac{s+1}{s^3+2s^2+Ks+K}$$

Making a Routh table:

s^3	1	K
s^2	2	K
s^1	$\frac{K}{2}$	0
s^0	K	0

Therefore, system is stable and steady-state error calculations are valid.

5.3. Steady-State Error for Non-Unity Feedback with Disturbance

Let us look at the general system of the figure below which has both a disturbance and non-unity feedback.

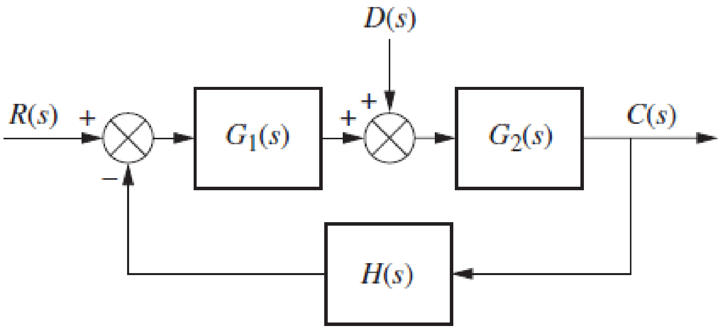


Figure 12: Non-unity feedback control system with disturbance

We will derive a general equation for the steady-state error and then determine the parameters of the system in order to drive the error to zero for step inputs and step disturbances.

The steady-state error for this system, $e(\infty) = r(\infty) - c(\infty)$, is:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left\{\left[1-\frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]R(s) - \left[\frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]D(s)\right\}$$

Now limiting the discussion to step inputs and step disturbances, where $R(s) = D(s) = 1/s$, the above equation becomes:

$$e(\infty) = \lim_{s\to 0} sE(s) = \left[1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right] - \left[\frac{\lim_{s\to 0} G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right]$$

For zero error,

$$\frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)} = 1 \qquad \text{and} \qquad \frac{\lim_{s\to 0} G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)} = 0$$

The two equations above can always be satisfied if:

(1) the system is stable,

(2) $G_1(s)$ is a Type 1 system,

(3) $G_2(s)$ is a Type 0 system, and

(4) $H(s)$ is a Type 0 system with a dc gain of unity.

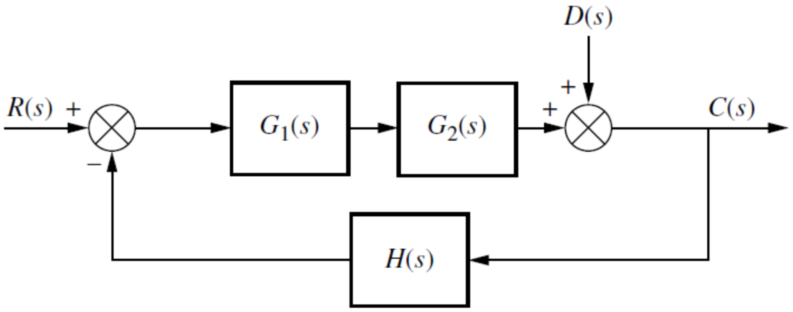
Example for Tutorial 9 – Non-Unity Feedback with Disturbance

Given the system shown in the figure below, do the following:

a. Derive the expression for the error, $E(s) = R(s) - C(s)$, in terms of $R(s)$ and $D(s)$. [8 marks]

b. Derive the steady-state error, $e(\infty)$, if $R(s)$ and $D(s)$ are unit step functions. [4 marks]

c. Determine the attributes of $G_1(s)$, $G_2(s)$, and $H(s)$ necessary for the steady-state error to become zero. [2 marks]



Answer

a. The error in the system is calculated from:

$$E(s) = R(s) - C(s)$$

But, taking into account the disturbance, the output of the system is:

$$C(s) = [R(s) - C(s)H(s)]G_1(s)G_2(s) + D(s)$$

Solving for $C(s)$:

$$C(s)=\frac{R(s)G_1(s)G_2(s)}{1+G_1(S)G_2(s)H(s)}+\frac{D(s)}{1+G_1(s)G_2(s)H(s)}$$

Substituting the above equation into $E(s)$, the equation becomes:

$$E(s)=\left[1-\frac{G_1(s)G_2(s)}{1+G_1(S)G_2(s)H(s)}\right]R(s)-\left[\frac{1}{1+G_1(s)G_2(s)H(s)}\right]D(s)$$

b. For $R(s)=D(s)=1/s$, the steady-state error of the system is:

$$e(\infty)=\lim_{s\to 0} sE(s)=1-\frac{\lim_{s\to 0} G_1(s)G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)}-\frac{1}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)}$$

c. Zero error if $G_1(s)$ and/or $G_2(s)$ is Type 1. Also, $H(s)$ is Type 0 with unity DC gain.

6. Sensitivity of System Parameters Towards Steady-State Errors

Sensitivity is the degree to which changes in system parameters affect system transfer functions, and hence performance. A system with zero sensitivity (that is, changes in the system parameters have no effect on the transfer function) is ideal. The greater the sensitivity, the less desirable the effect of a parameter change.

For example, assume the function of:

$$F=\frac{K}{(K+a)}$$

If K = 10 and a = 100, then F = 0.091. If parameter a triples to 300, then F = 0.032. We see that a fractional change in parameter a of $(300-100)/100=2$ (e.g. 200% change) yields a change in the function F of $(0.032-0.091)/0.091=0.65$ (e.g. 65% change). Thus, the function F has reduced sensitivity to changes in parameter a.

As we proceed, we will see that another advantage of feedback is that in general it affords reduced sensitivity to parameter changes. Based upon the discussion, given above formal definition of sensitivity is the ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero. That is,

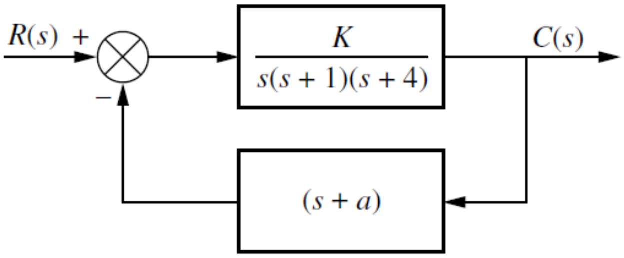
$$S_{F:P}=\lim_{\Delta P\to 0}\frac{\text{Fractional change in the function, }F}{\text{Fractional change in the parameter, }P}=\lim_{\Delta P\to 0}\frac{\Delta F/F}{\Delta P/P}=\lim_{\Delta P\to 0}\frac{P\Delta F}{F\Delta P}$$

Which reduces to:

$$S_{F:P}=\frac{P}{F}\left(\frac{\delta F}{\delta P}\right)$$

Example of Tutorial 10 – Sensitivity of System Parameters

For a system as shown in the figure below, assume it is given a step input.



a. Find the sensitivity of the steady-state error to parameter a. [6 marks]

b. Plot the sensitivity of the system as a function of parameter a. [5 marks]

Answer

a. First, find the forward transfer function of an equivalent unity-feedback system.

$$G_e(s)=\frac{\frac{K}{s(s+1)(s+4)}}{1+\frac{K(s+a-1)}{s(s+1)(s+4)}}=\frac{K}{s^3+5s^2+(K+4)s+K(a-1)}$$

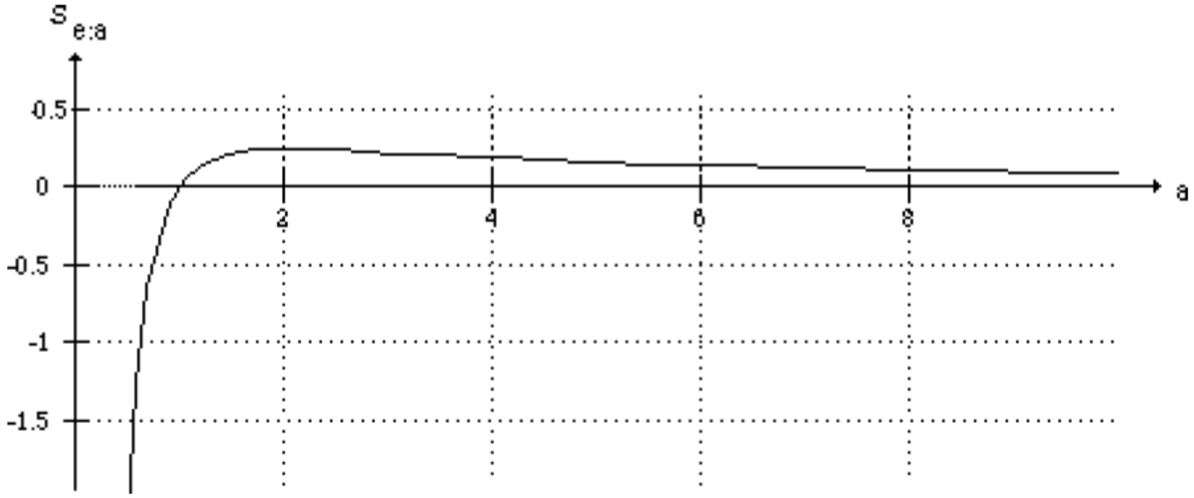
Thus, steady-state error of the system is:

$$e(\infty)=\frac{1}{1+K_p}=\frac{1}{a+\frac{K}{K(a-1)}}=\frac{a-1}{a}$$

Finding the sensitivity of $e(\infty)$, it is:

$$S_{e:a}=\frac{a}{e}\left(\frac{\delta e}{\delta a}\right)=\frac{a}{\left(\frac{a}{a-1}\right)}\left[\frac{a-(a-1)}{a^2}\right]=\frac{a-1}{a^2}$$

b. The plot of sensitivity of the system as a function of parameter a is as shown in the figure below.


S e:a
0.5
0
-0.5
-1
-1.5
2
4
6
8
a

Appendix: Steady-State Analysis of Control Systems

Input Name	**Input Parameters**	**Steady-state Error Formula**	**Type 0**		**Type 1**		**Type 2**	
			Static Error Constant	**Error**	**Static Error Constant**	**Error**	**Static Error Constant**	**Error**
Step	$u(t)$	$\frac{1}{1+K_p}$	K_p = Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp	$tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	K_v = Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	K_a = Constant	$\frac{1}{K_a}$