XMUT315 Control Systems Engineering

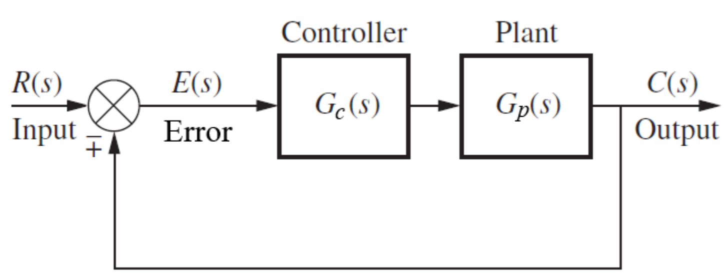
Note 9a: Introduction to Controllers and Compensators

Topics:

- Controller and Compensator in the Feedback Control Systems.
- Controllers' Characteristics (e.g. Proportional Controller, Proportional-Integral Controller, Proportional-Derivative Controller, and Proportional-Derivative-Integral Controller).
- Compensators' Characteristics (e.g. Lead Compensator, Lag Compensator, and Lead-Lag Compensator).
- Intro to Compensator Design.

1. Introduction to Controller and Compensator

A controller is an element whose role is to maintain a physical quantity at a desired level. On the other hand, the compensator is an element for modification of system dynamics, to improve characteristics of the open-loop plant so that it can safely be used with feedback control.



Note: G_c = controller or compensator and G_p = plant.

Figure 1: Controller or compensator in the feedback control systems

Three main types of controllers and their practical combinations:

- P (Gain or Proportional) controller.

- I (Integral) controller
- D (Derivative) controller
- PD (Proportional-Derivative) controller
- PI (or Proportional Integral) controller
- PID (Proportional, Integral, and Derivative) controller.

Three main types of compensators:

- Lag.
- Lead.
- Lead-lag.

When observing the influence of controllers or compensators on the control systems:

- They change the natural response of the system.
- They adjust the poles of the system.
- They help achieve the desired output from a given input.

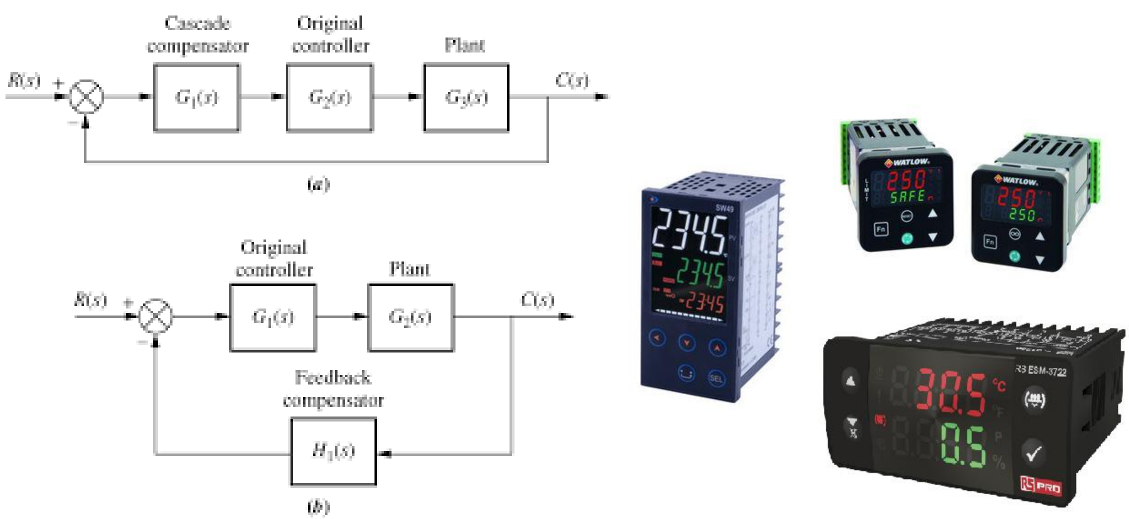


Figure 2: Controllers and compensators: abstract (left) and practical representations (right)

2. Controllers

In this section, we look into more detailed coverage of the controllers as stated above.

Based on its purpose, controllers are typically implemented to control or manage the control systems. This is different from the compensators that are commonly used to attend a specific issue or problem in the control systems.

Controllers for electronic systems are commonly constructed using active components such as op amp. These active components are needed due to the amplification nature of the proportional controller and integration and differentiation operations of the integral and differential controllers.

2.1. P Controller

Given a gain or proportional controller ($G_c(s)$) implemented in series with the plant ($G(s)$) in a control system as shown in the figure below.

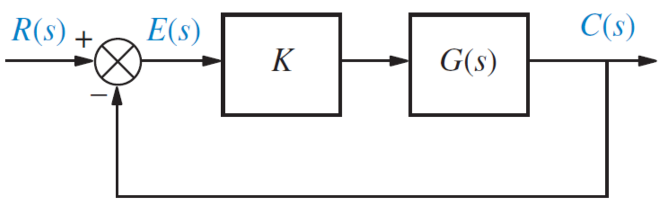


Figure 3: Proportional controller in the control system

The transfer-function equation of the proportional controller is:

$$G_c(s) = K$$

Thus, the transfer-function equation of the closed-loop system with a series proportional controller is:

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{KG(s)}{1 + KG(s)}$$

For a proportional controller ($G_c(s)$) attached in the feedback loop, consider a plant in the forward path ($G(s)$).

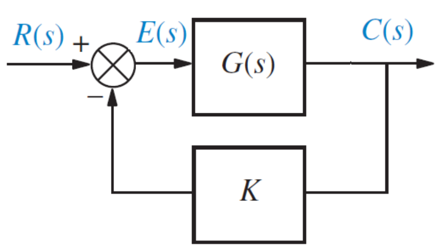


Figure 4: Feedback control systems

The transfer-function equation of the closed-loop feedback system with a proportional controller at the feedback path is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - (-G_c(s)G(s))} = \frac{G(s)}{1 + KG(s)}$$

Notice the negative sign in the transfer-function equation of the feedback system. If the size of the loop gain is large, that is if:

$$|KG(s)| \gg 1$$

Then, the gain of the closed-loop system approximately depends on the gain of the controller:

$$T(s) \approx \frac{G(s)}{KG(s)} = \frac{1}{K}$$

2.1.1. Characteristics of P Controllers

The frequency response of a P controller is as shown below.

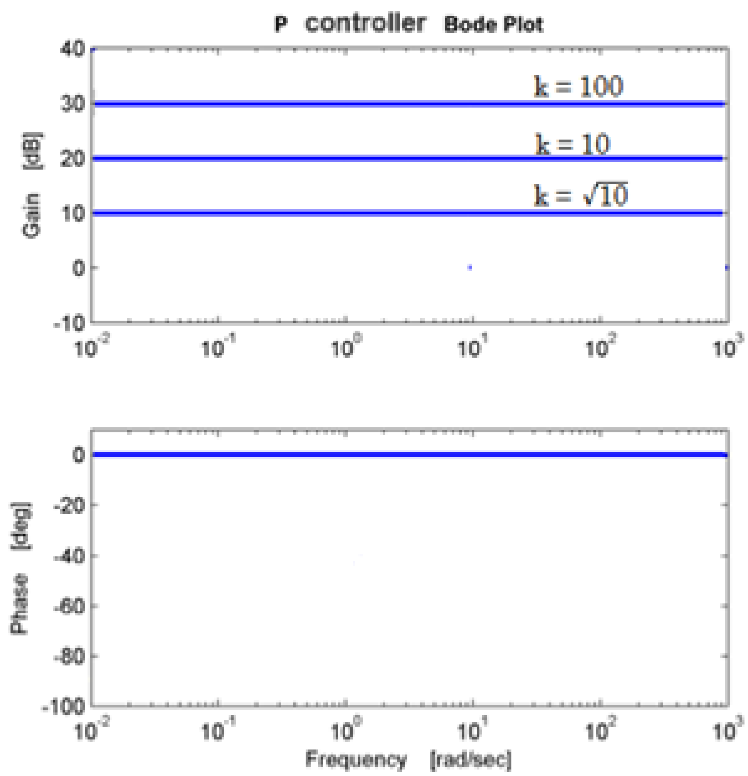


Figure 5: Frequency response of P controller

- Magnitude plot:
    - All frequency: $20 \log K$
- Phase-shift plot:
    - All frequency: 0°

With these characteristics given above, the P controller is often used to improve transient response (up to a point). It can increase the gain of the system and often result in a non-zero steady-state error. Practically, it is relatively easy to implement.

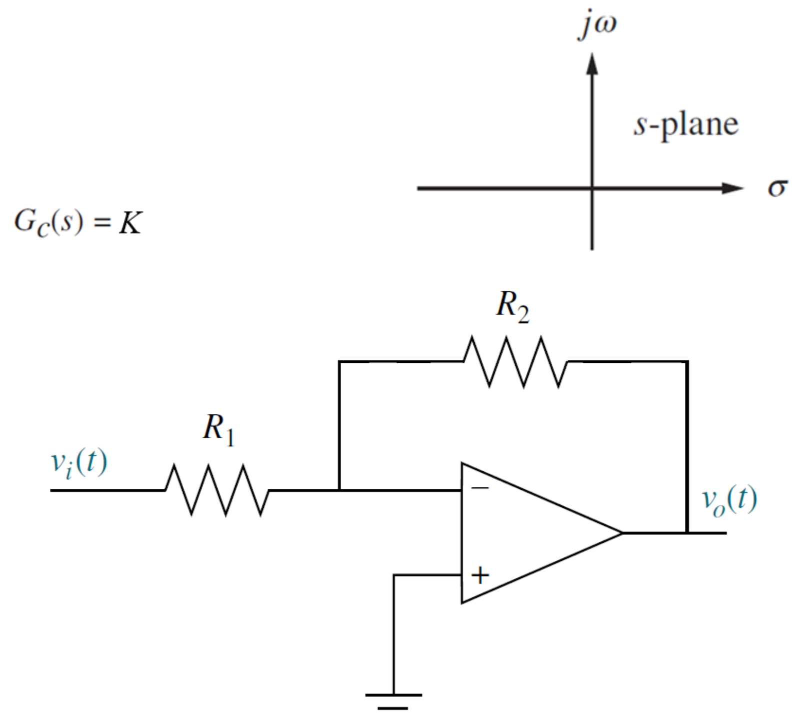


Figure 6: P-controller poles and zeros and its circuit implementation

2.1.2. Application of P Controller in First-Order System

For a proportional controller ($G_c(s)$) is attached in the feedback loop as shown in the figure below, the plant in the forward path is a first-order system $G(s)$.

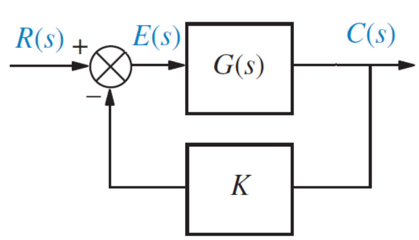


Figure 7: Proportional controller in first-order system

The transfer-function equations of both the controller and the plant are:

$$G_c(s) = K_p \qquad \text{and} \qquad G(s) = \left(\frac{A}{1 + sT}\right)$$

In term of $T(s) = C(s)/R(s)$, closed-loop transfer-function equation of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)}{1 - \big(-G_c(s)\big)G(s)}$$

$$= \frac{\left(\frac{A}{1+sT}\right)}{1 - (-K)\left(\frac{A}{1+sT}\right)} = \frac{A}{1 + sT + AK}$$

Thus, we can see that the time constant of the closed-loop first-order system depends on both gains of the plant A and controller K. The gain of the closed-loop system depends on the gain of the plant A.

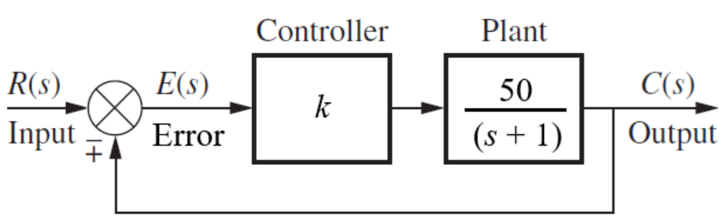
Example for Tutorial 1: P-Controller for First-Order System

The open-loop transfer-function equation of a first-order system is given below.

$$G(s) = \frac{50}{s+1} \qquad \text{and} \qquad G_C(s) = K$$

a. Determine the time constant of the open loop system. [2 marks]

b. If a proportional controller connected in series with the system as shown below, determine the gain of proportional controller (K) that will change the time constant (τ) of the closed-loop system to become 0.1 second. [6 marks]



Answer

a. The time constant of the open-loop first-order system is:

$$G(s) = \frac{50K}{s+a}$$

Thus

$$\tau = \frac{1}{a} = \frac{1}{1} = 1\ \text{s}$$

b. The gain of the proportional controller (K) that will change the time constant of the closed-loop first-order system to become 0.1 second is determined as follows.

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K\left(\frac{50}{s+1}\right)}{1 + K\left(\frac{50}{s+1}\right)} = \frac{50K}{s + 1 + 50K}$$

Thus

$$\tau = \frac{1}{a} = \frac{1}{1 + 50K}$$

For the time constant of 0.1 second, the gain of the proportional controller is calculated from:

$$0.1 = \frac{1}{1 + 50K}$$

Rearrange the equation above, the value of K is:

$$K = \frac{10 - 1}{50} = 0.18$$

2.1.3. P Controller in Second-Order Systems

There are several further applications of P controller in the control systems, especially for higher order systems. These are described in the following sections.

2.1.3.1. Second-Order Systems

For a second-order system (i.e. $G(s)$) with a proportional controller ($H(s)$) added as shown below.

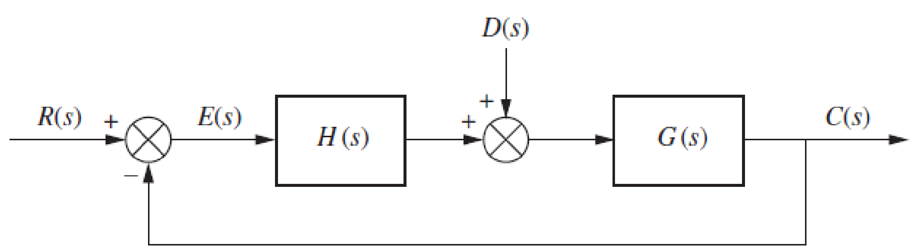


Figure 8: P controller in a second-order system with unity feedback

Unless told otherwise, assume $D(s)$ = 0. Thus, the transfer-function equation of unity feedback second-order system with P controller is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

Considering $G(s) = \omega_n^2/(s^2 + 2\omega_n\zeta s + \omega_n^2)$ and $H(s) = K$, then the equation above becomes:

$$T(s) = \frac{\left[\frac{\omega_n^2}{(s^2 + 2\omega_n\zeta s + \omega_n^2)}\right]K}{1 + \left[\frac{\omega_n^2}{(s^2 + 2\omega_n\zeta s + \omega_n^2)}\right]K} = \frac{\omega_n^2 K}{s^2 + 2\omega_n\zeta s + (1 + K)\omega_n^2}$$

Considering the characteristics equation, determine values for $1 + K$ to make the system:

- Undamped
- Overdamped
- Critically damped
- Overdamped

Higher gain is typically yielding a faster response, but it is at the expense of a more oscillatory response.

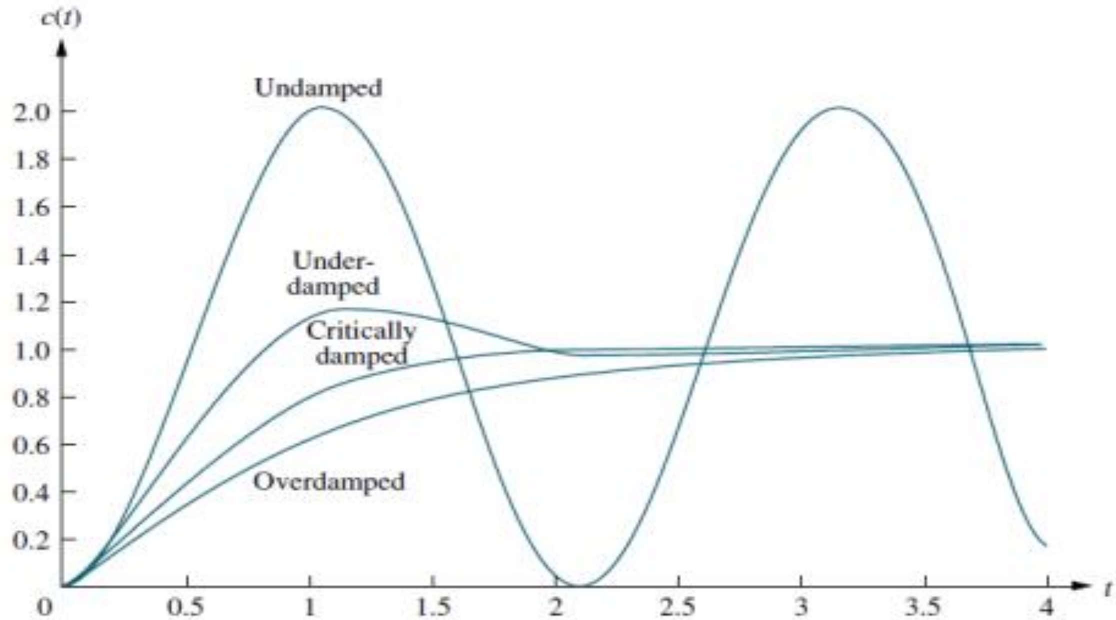


Figure 9: Time responses of the second-order system

As proven for the first-order system before, the higher gain is typically yielding a faster response, but it is at the expense of a more oscillatory response in the second-order system.

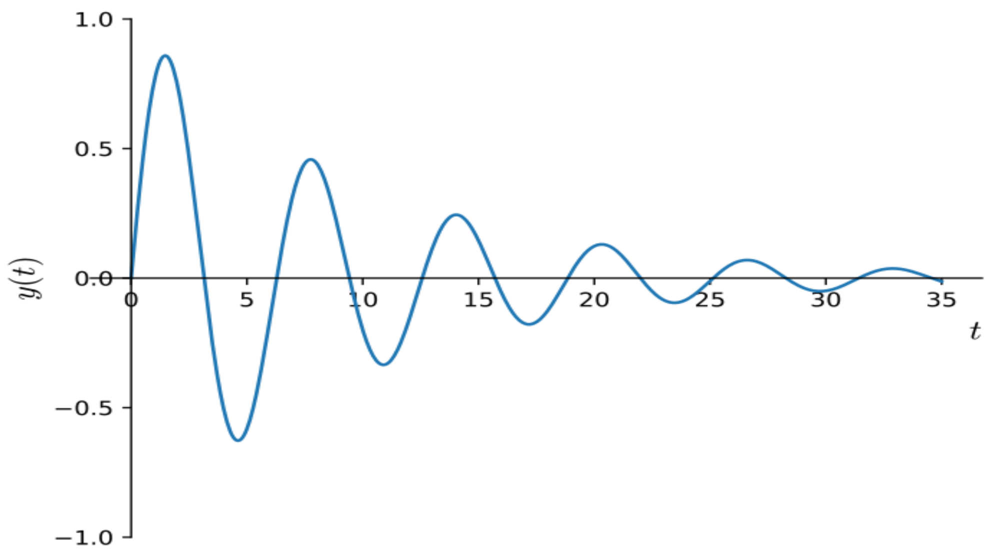


Figure 10: Damped oscillatory response of a second-order system

If the transient response of the system is too oscillatory, it will take time before the system settles to its final value. We cannot, therefore, just increase the controller gain.

2.1.3.2. Second-Order Systems, Non-Unity Feedback

For a second-order system with a proportional controller ($M(s)$) in non-unity feedback as shown below.

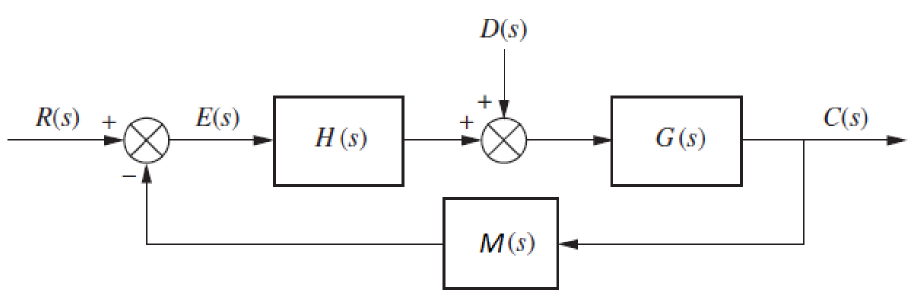


Figure 11: P controller in a non-unity feedback second-order system

If $D(s) = 0$, the gain of the open-loop system is:

$$H(s)G(s)$$

The transfer-function equation of the closed-loop system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{H(s)G(s)}{1 + M(s)H(s)G(s)}$$

Focusing on the characteristic equation of the transfer-function equation, it is:

$$1 + M(s)H(s)G(s)$$

Notice that $M(s)$ influences the characteristic equation. The variable $M(s)$ affects transient response of the closed-loop system as specified above.

Applying final-value theorem, the steady-state equation for the output of the system for a step input is:

$$C(\infty) = \lim_{s\to 0} s\,(R(s))\frac{H(s)G(s)}{1 + M(s)H(s)G(s)} = \lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{H(s)G(s)}{1 + M(s)H(s)G(s)}\right) \cong \frac{1}{M(s)}$$

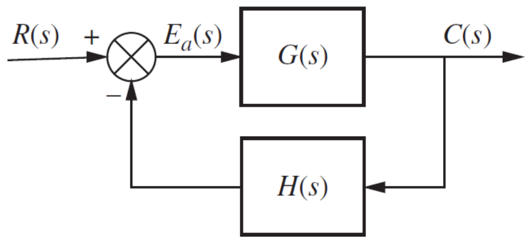
Thus, the variable $M(s)$ influences also the steady-state response of the closed-loop system for a given step input.

Example for Tutorial 2: P-Controller in Second-Order System

For an open-loop control system described as the transfer-function equation given below, attempt the following tasks.

$$G(s) = \frac{5}{s^2 + 10s + 5}$$

a. Derive the transfer-function equation of the closed-loop system is a proportional controller $H(s) = M$ is added in the feedback loop as shown below. [4 marks]



b. If M is 9, determine the transient response of the closed-loop system. [6 marks]

c. As part of design specification for the system, for a step input response, determine the feedback gain (M) if we wish the steady-state error condition of the closed-loop system to be 0.6. [8 marks]

Answer

a. For the given second-order system with non-unity feedback, the transfer-function equation of the closed-loop system is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\frac{5}{s^2 + 10s + 5}}{1 + \left(\frac{5}{s^2 + 10s + 5}\right) M} = \frac{5}{s^2 + 10s + 5(1 + M)}$$

b. The transient response of the closed-loop system when M is 9 is:

$$T(s) = \frac{5}{s^2 + 10s + 5(1 + 9)} = \frac{5}{s^2 + 10s + 50}$$

Evaluating the characteristics equation of the closed-loop system, its roots are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(50)}}{2(1)} = -5 \pm j5$$

The roots are complex pair, so the response of the closed-loop system is underdamped.

c. For a step input, the steady-state error of the closed-loop system is:

$$e_{step}(\infty) = \lim_{s \to 0} \frac{s\left(\frac{1}{s}\right)}{1 + T(s)}$$

$$= \lim_{s \to 0} \frac{1}{1 + \frac{5}{s^2 + 10s + 5(1 + M)}}$$

$$= \lim_{s \to 0} \frac{s^2 + 10s + 5(1 + M)}{s^2 + 10s + 5(2 + M)} = \frac{1 + M}{2 + M}$$

To achieve a steady-state error of 0.6 for a step response of the system, the gain of proportional controller M is calculated from:

$$e_{step}(\infty) = \frac{1 + M}{2 + M} = 0.6$$

Thus, the gain of the proportional controller that meets the design specification is:

$$M = \frac{1.2 - 1}{1 - 0.6} = 0.5$$

2.1.4. P Controller in Practice

In practice, the P controller is realised as a non-inverting amplifier with R_1 and R_2 forming the voltage divider part of the circuit.

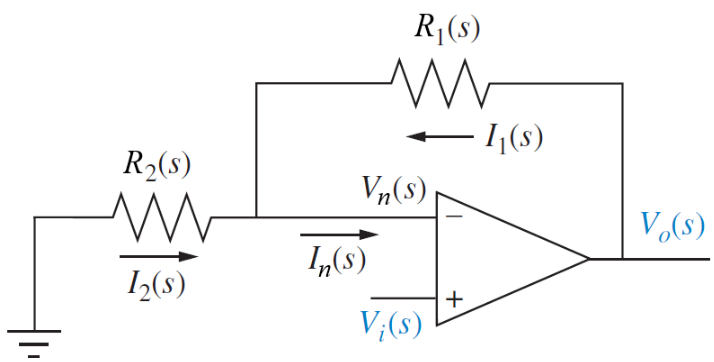


Figure 12: Non-inverting op amp amplifier circuit for realising P controller.

The voltage at the non-inverting input is:

$$V_p = V_{IN}$$

Due to potential divider arrangement in the circuit, the voltage at the inverting pin of the op amp is:

$$V_n = V_{OUT}\left(\frac{R_2}{R_1 + R_2}\right)$$

As an example, consider a non-inverting amplifier with an open-loop gain of $A_0(s)$ as shown in the figure below.

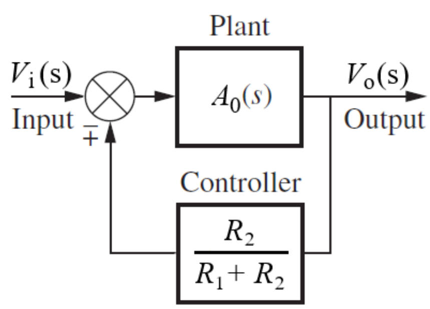


Figure 13: P controller circuit in its block diagram

The transfer-function equation of the amplifier is:

$$\frac{V_O(s)}{V_I(s)} = \frac{A_0(s)}{1 - (-A)\left(\frac{R_2}{R_1 + R_2}\right)} = \frac{A_0(s)(R_1 + R_2)}{R_1 + R_2 + AR_2}$$

If the loop gain $A_0(s)R_2/(R_1 + R_2)$ is large, $A_0(s)R_2 \gg R_1 + R_2$:

$$\frac{V_O(s)}{V_I(s)} = \frac{A_0(s)(R_1 + R_2)}{A_0(s)R_2} = \frac{R_1 + R_2}{R_2}$$

In the above equation, feedback gain, $\beta = R_2/(R_1 + R_2)$, so if loop gain is large:

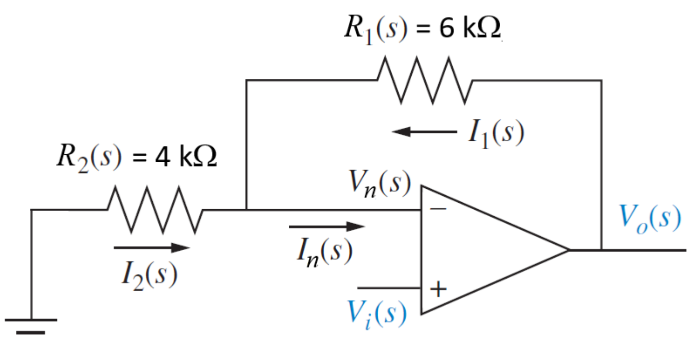
$$\frac{V_O(s)}{V_I(s)} = \frac{1}{\beta} = \frac{R_1 + R_2}{R_2}$$

Example for Tutorial 3: Practical Implementation of P-Controller

For example, given design specification of a non-inverting operational amplifier circuit as shown below, perform the following tasks:

Note:

- Open-loop gain of the op amp, $A_0(s)$ = 10^5.
- Feedback resistors: R_1 = 6 kΩ and R_2 = 4 kΩ.



a. Derive the transfer-function equation of the amplifier. [6 marks]

b. Determine whether the forward-loop gain of the amplifier is larger than the feedback-loop gain. [4 marks]

c. Determine the gain of the amplifier. [4 marks]

Answer

a. The transfer-function equation of the operational amplifier circuit is derived from:

$$V_p = V_i \qquad (Eq.1)$$

And

$$V_n = V_o\left(\frac{R_2}{R_1 + R_2}\right) = V_O\left(\frac{4\text{k}\Omega}{4\text{k}\Omega + 6\text{k}\Omega}\right) = 0.4V_O \qquad (Eq.2)$$

For a given non-inverting amplifier with an open-loop gain of A, the output voltage is:

$$V_o = A_0(s)\big(V_p - V_n\big) = A_0(s)(V_I - 0.4V_O) \qquad (Eq.3)$$

Substituting equations (1) and (2) into equation (3), the transfer function equation of the op amp circuit is:

$$\frac{V_o}{V_i} = \frac{A_0(s)}{1 + 0.4A_0(s)}$$

b. For the given operational amplifier circuit, the forward-loop gain of the amplifier is:

$$A_0(s)R_2 = (10^5)(4 \times 10^3) = 4 \times 10^8$$

The feedback-loop gain of the amplifier is:

$$R_1 + R_2 = 6 \times 10^3 + 4 \times 10^3 = 10^4$$

As a result, the forward-loop gain of the amplifier is larger than the feedback-loop gain of the amplifier:

$$A_0(s)R_2 \gg R_1 + R_2$$

c. Thus, for a non-inverting amplifier circuit, the gain of the amplifier is calculated from:

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1 + R_2}{R_2} = \frac{6 \times 10^3 + 4 \times 10^3}{4 \times 10^3} = 2.5$$

2.2. PI Controllers

Given a PI controller in the control system as shown in the figure below.

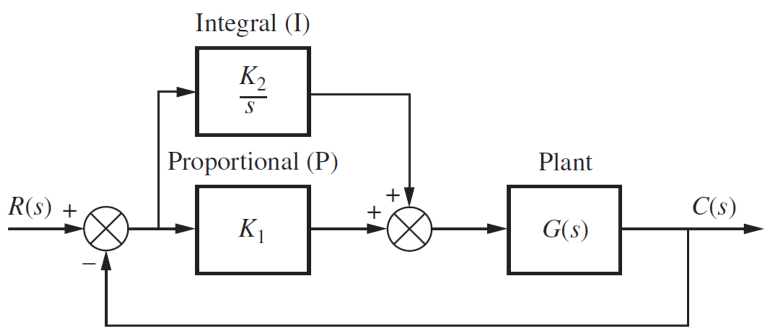


Figure 14: PI controller in the control system

For a PI Controller, its transfer-function equation can be written as:

$$G_c(s) = P(s) + I(s) = K_1 + \frac{K_2}{s} = \frac{K_1\left(s + \frac{K_2}{K_1}\right)}{s}$$

Here, $P(s) = K_1$ and $I(s) = K_2/s$.

2.2.1. Characteristics of PI Controllers

The frequency response of a PI controller is as shown below.

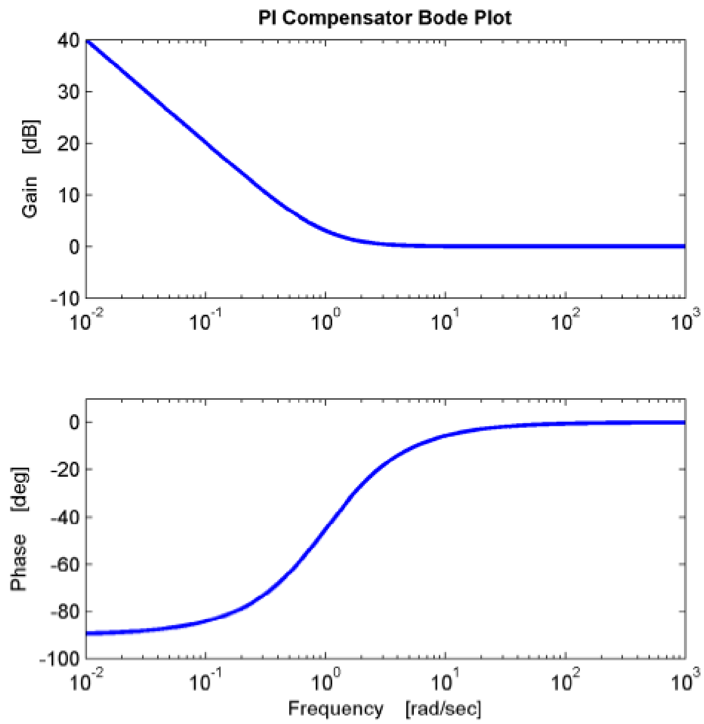


Figure 15: Frequency response of PI controller

- Magnitude plot:
  - Low: -slope gain.
  - Cut-off: half gain.
  - High: zero gain.
- Phase-shift plot:
  - Low: -90°.
  - Cut-off: -45°.
  - High: 0°.

With these characteristics, PI controller is used for improving steady-state error. It increases system type and hence steady-state error becomes zero. In this controller, zero at z_c is small and negative. In practice, this controller needs active circuits to implement.

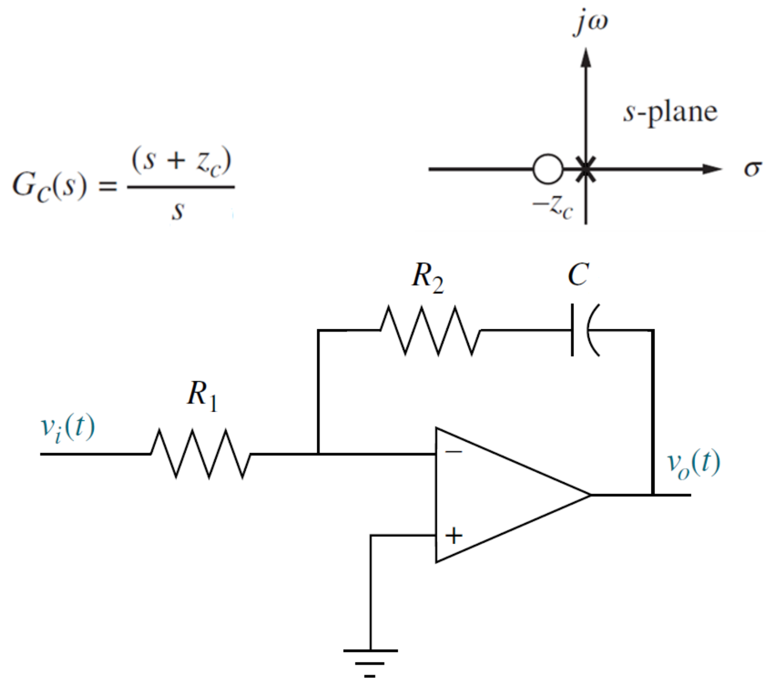


Figure 16: PI controller poles and zero and its circuit implementation

2.2.2. Applications of PI Controllers

The functions $P(s)$ and $I(s)$ can be chosen so the $s + K_2/K_1$ term (controller zero) cancels plant pole. Suppose a plant of a second-order system is:

$$G(s) = \frac{1}{(1 + sT_1)(1 + sT_2)}$$

If we apply PI to this plant, and make $K_2\,/K_1 = T_2$, then

$$\frac{O(s)}{E(s)} = G_c(s)G(s) = \left(\frac{s + K_1/K_2}{s}\right)\frac{1}{(1 + sT_1)(1 + sT_2)} = \frac{1}{s(s + T_1)}$$

For a closed-loop system, the transfer-function equation is:

$$T(s) = \frac{O(s)}{I(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{\frac{1}{s(s + T_1)}}{1 + \frac{1}{s(s + T_1)}} = \frac{1}{s^2 + sT_1 + 1}$$

Note, the $I(s)$ term means that the steady-state value is 1.

2.3. PD Controllers

For a PD controller in a given control system as shown in the figure below.

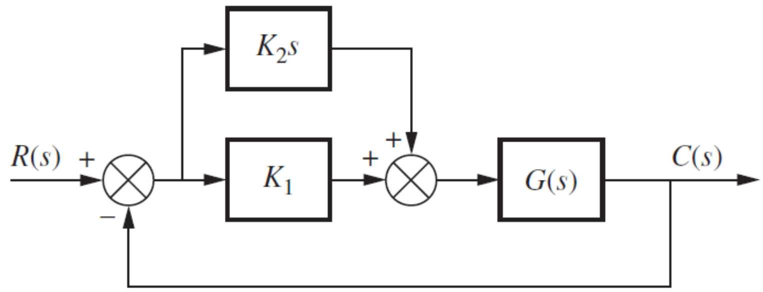


Figure 17: PD controller in the control system

For a PD controller, its transfer-function equation is:

$$G_1(s) = P(s) + D(s) = K_1 + K_2 s = K_2(s + K_1/K_2)$$

Where: $P(s) = K_1$ and $D(s) = K_2 s$.

2.3.1. Characteristics of PD Controllers

The frequency response of a PD controller is as shown below.

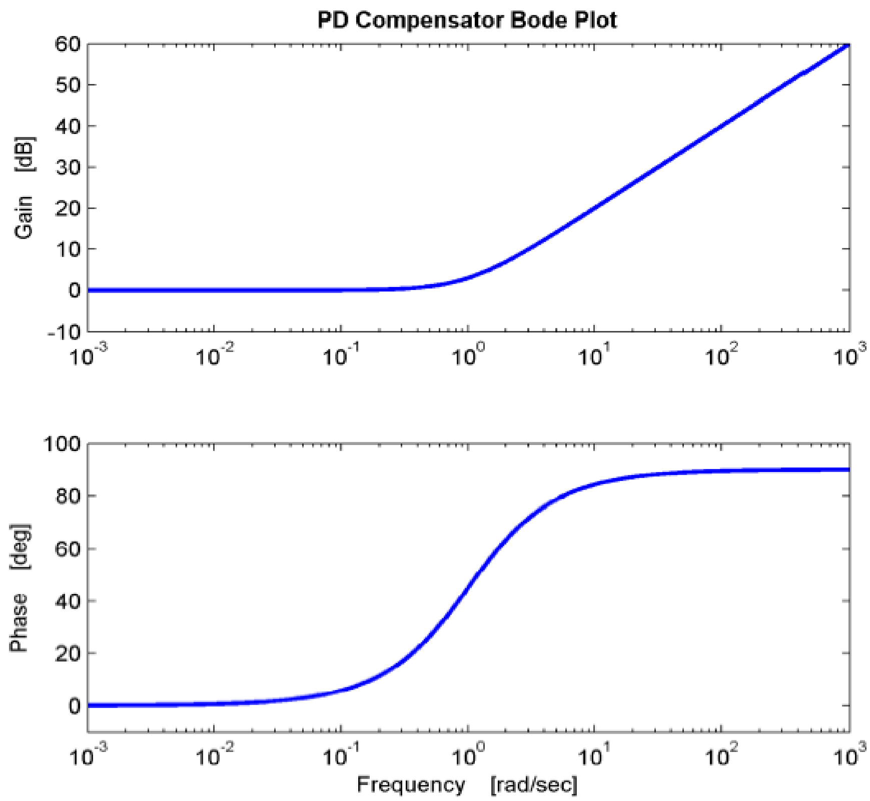


Figure 18: Frequency response of PD controller

- Magnitude plot:
  - Low: zero gain.
  - Cut-off: half gain.
  - High: +slope gain.
- Phase-shift plot:
  - Low: 0°.
  - Cut-off: +45°.
  - High: +90°.

The PD controller is often used to improve transient response. In its design, its zero at $-z_c$ is selected to indicate the design point. It requires active circuits to implement; implemented with rate feedback or with a pole (lead). Furthermore, this controller can cause noise and saturation.

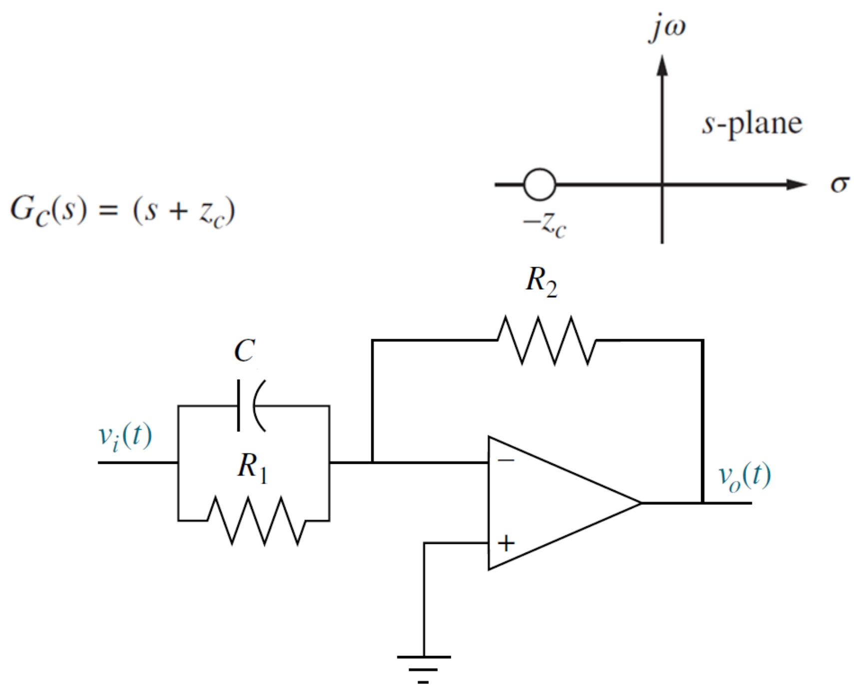


Figure 19: PD controller poles and zeros and its circuit implementation

2.3.2. Applications of PD Controllers

We could make $s + K_1/K_2$ term to cancel the plant pole. If the transfer-function equation of the plant of a second-order system is:

$$G(s) = \frac{1}{s(s+T)}$$

If PD is applied, time constant $K_1K_2 = T$ then:

$$\frac{O(s)}{E(s)} = G_1(s)G_2(s) = (1+sT)\frac{1}{s(1+sT)} = \frac{1}{s}$$

The transfer-function equation of the closed-loop system is:

$$T(s) = \frac{O(s)}{I(s)} = \frac{1/s}{1+1/s} = \frac{1}{s+1}$$

2.4. PID Controllers

For a PID controller as shown in the figure below

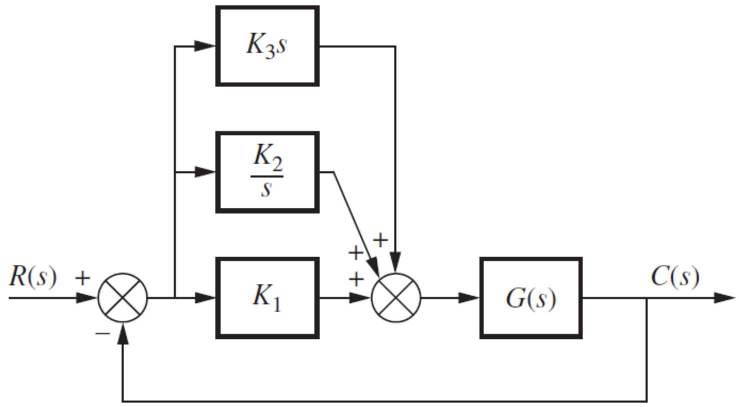


Figure 20: PID controller in the control system

For a PID controller, its transfer-function equation is:

$$G_c(s) = P(s) + I(s) + D(s) = K_1 + \frac{K_2}{s} + K_3 s$$

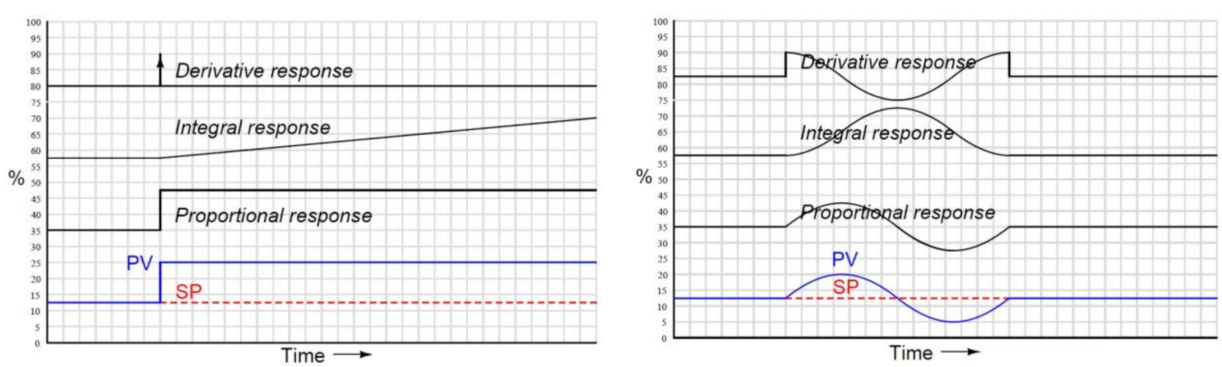
Where: $P(s) = K_1$, $I(s) = K_2/s$, and $D(s) = K_3 s$

This gives:

$$G_c(s) = \frac{K_3 s^2 + K_1 s + K_2}{s}$$

2.4.1. Characteristics of PID Controllers

The responses of the PID controllers against the step and sinusoidal inputs are as shown in the figure below.



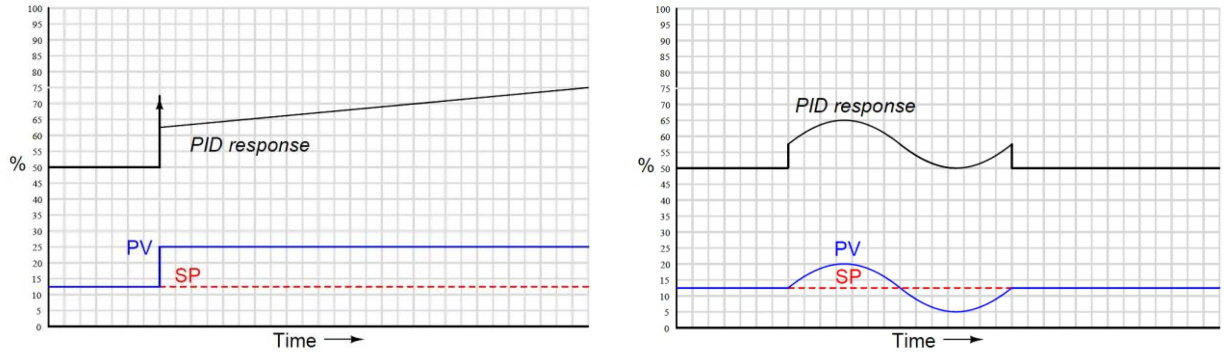


Figure 21: Response of PID controllers with the step and sinusoidal inputs

The PID controller is used to improve both steady-state error and transient response. In its design, its lag zero at $-z_{lag}$ and pole at the origin improve steady-state error. Its lead zero at $-z_{lead}$ improves transient response. The lag zero at $-z_{lag}$ is close to, and to the left of, the origin and the lead zero at $-z_{lead}$ is selected to indicate the design point. Its implementation requires active circuits; implemented with rate feedback or with an additional pole. It can cause noise and saturation.

$$G_C(s) = \frac{(s + z_c)(s + z_c')}{s}$$

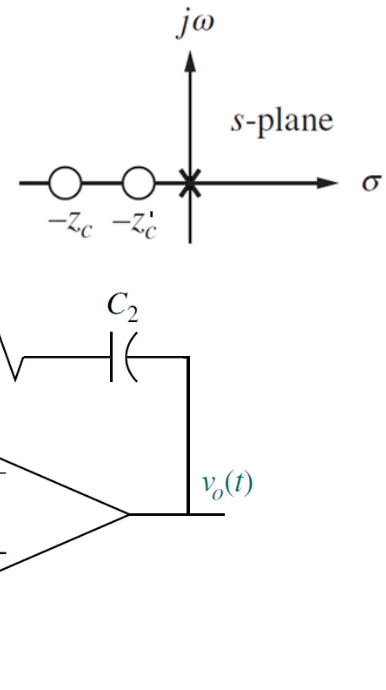


Figure 22: PID controller poles and zeros and its circuit implementation

2.4.2. Applications of PID Controllers

For a given second-order system with two poles, its transfer-function equation is:

$$G(s) = \frac{1}{1 + bs + as^2}$$

If $K_3 = a$ and $K_1 = b$, and $K_2 = 1$, then

$$T(s) = \frac{O(s)}{E(s)} = \left(\frac{K_3 s^2 + K_1 s + K_2}{s}\right)\left(\frac{1}{1 + bs + as^2}\right) = G_c(s)G(s) = \frac{1}{s}$$

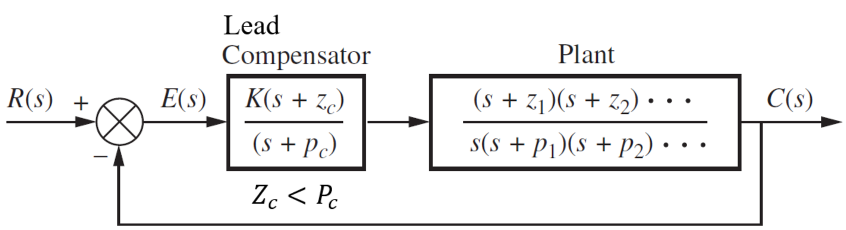
In all these examples, by careful arrangement, systems are first or second order. Cancellation may not give the best response, but analysis of systems is easier!

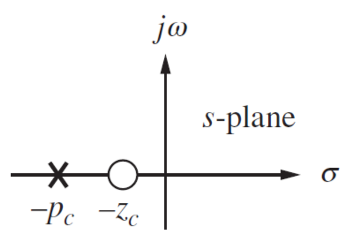
3. Compensators

Compared with the controllers, compensators are different in terms of their purposes and constructions. They are intended to be used for specific area of control system improvement and solution. Typically, they are made up of passive components in electronic systems.

3.1. Lead Compensator

For a lead compensator as shown in the figure below, the zero is located near y-axis compared with its pole i.e. $Z_c < P_c$.





$$G_c(s) = \frac{(s + z_c)}{(s + p_c)}$$

Figure 23: Lead compensator in the control system

The transfer-function equation of a lead compensator is:

$$G_c(s) = \frac{1}{\beta}\left(\frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}\right) \qquad (\beta < 1)$$

Or

$$G_{lead}(s) = \frac{s + z_c}{s + p_c} \qquad \text{with} \qquad |p_c| > |z_c|$$

This consists of 1 pole and 1 zero with the magnitude of the pole is bigger than the magnitude of the zero.

3.1.1. Characteristics of Lead Compensators

Frequency response plot of lead compensator is as shown in the figure below.

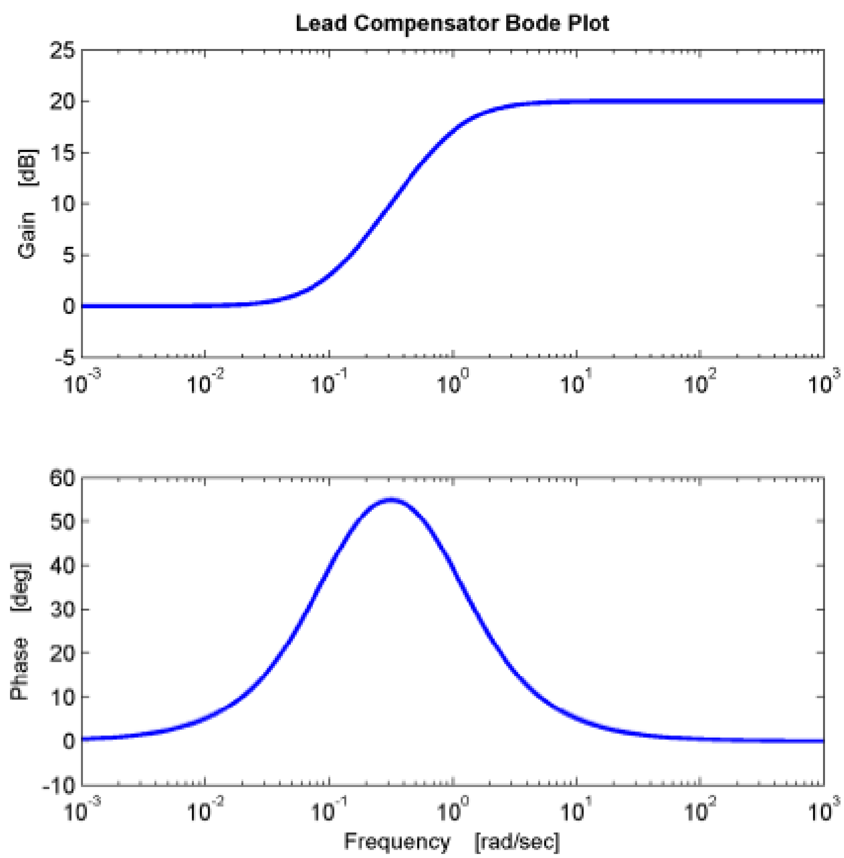


Figure 24: Frequency response of lead compensator

- Magnitude plot:
    - Low: zero gain.
    - Cut-off: half gain.
    - High: +finite gain.

- Phase-shift plot:
  - Low: 0°.
  - Cut-off: +45°.
  - High: 0°.

The lead compensator is used to improve the transient response. Its design typically places zero at $-z_c$ and pole at $-p_c$ at are selected to indicate design point. Its pole at $-p_c$ is more negative than zero at $-z_c$. Active circuits are not required to implement this compensator.

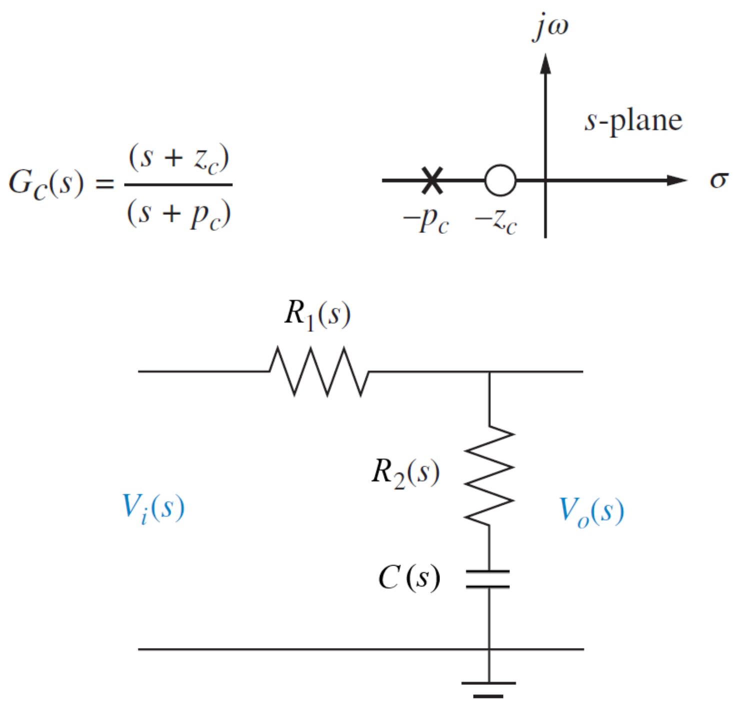


Figure 25: Lead compensator poles and zeros and its circuit implementation

The following figure shows frequency response simulation in MATLAB of lead compensator with various values of β.

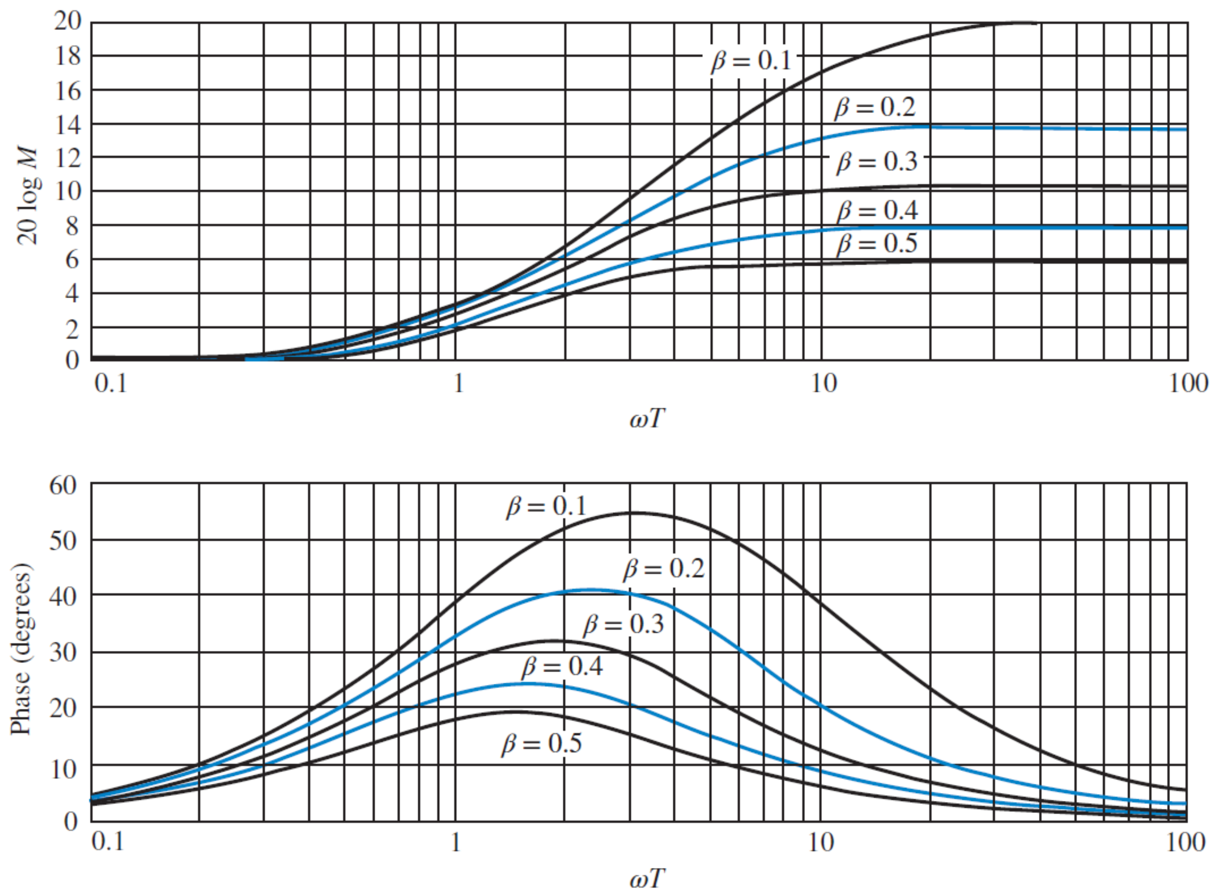


Figure 26: Frequency response of lead compensator with various values of β

3.1.2. Applications of Lead Compensators

In lead compensator, the zero is closer to the origin than the pole, that is:

$$z_c < p_c$$

The lead compensator influences transient response (e.g. % overshoot and settling times).

The following diagram compares the transient responses of the uncompensated system with the compensated system. The lead compensator influences transient response (e.g. percentage overshoot and settling times) with a, b, and c = increasing distance of the poles from origin.

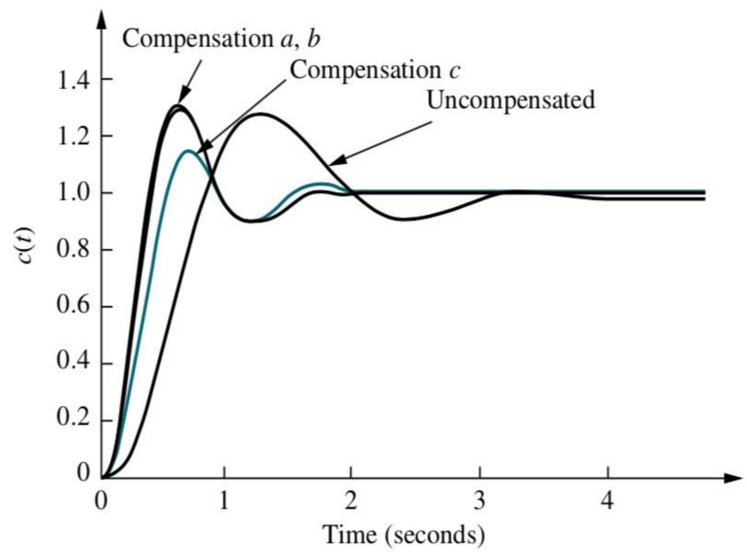


Figure 27: Transient responses of compensated and uncompensated systems

3.2. Lag Compensators

For a lag compensator as shown in the figure below, the pole is located closer to y- axis than its zero i.e. $Z_c > P_c$.

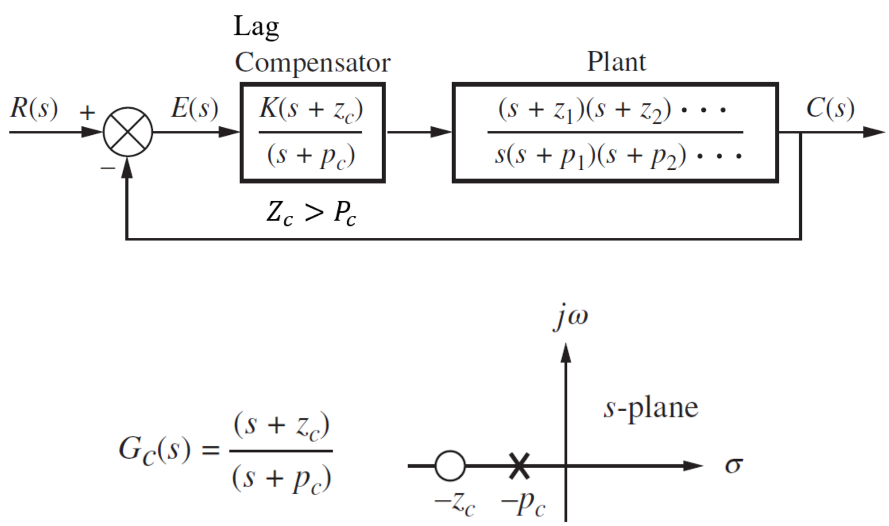


Figure 28: Lag compensator in the control system

The transfer-function equation of a lag compensator is:

$$G_c(s) = \left(\frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \right) \qquad (\alpha > 1)$$

Or

$$G_{lag}(s) = \frac{s + z_c}{s + p_c} \quad \text{with} \quad |p_c| < |z_c|$$

This compensator consists of 1 pole and 1 zero with the magnitude of the pole is smaller than the magnitude of the zero.

3.2.1. Characteristics of Lag Compensators

Frequency response plot of lag compensator is as shown in the figure below.

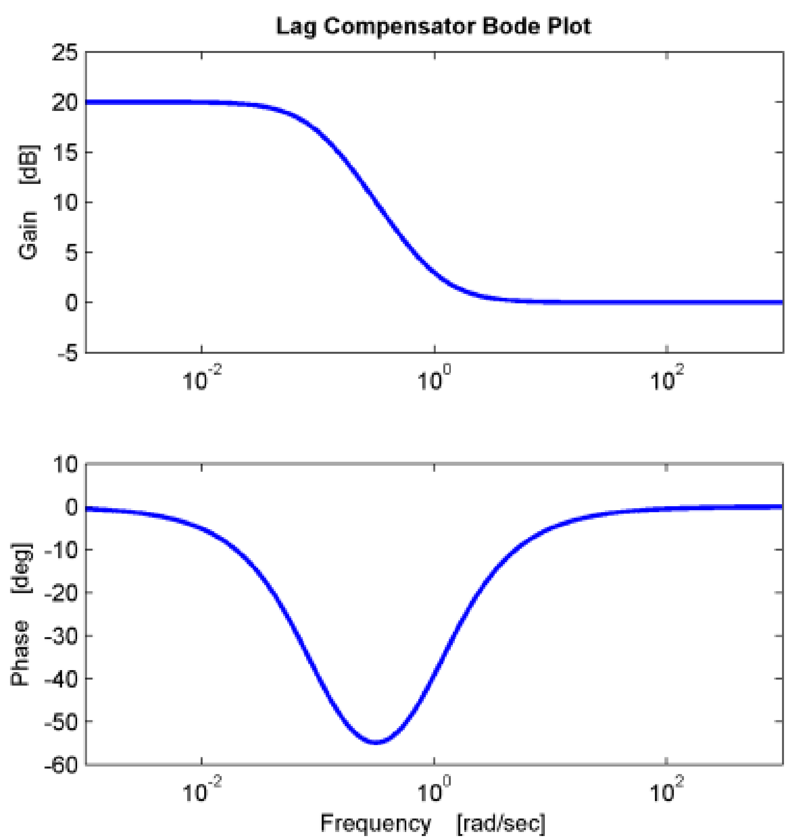


Figure 29: Frequency response of lag compensator

- Magnitude plot:
    - Low: +finite gain.
    - Cut-off: half gain.

    - High: zero gain.
- Phase-shift plot:
    - Low: 0°.
    - Cut-off: -45°.
    - High: 0°.

The lag compensator is used to improve the steady-state error. When implementing this controller in the system, the error is improved but not driven to zero. Its pole at $-p_c$ is small and negative and its zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. Active circuits are not required to implement this compensator.

$$G_C(s) = \frac{(s + z_c)}{(s + p_c)}$$

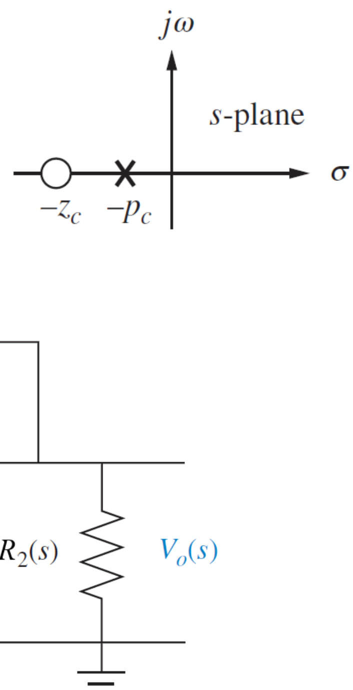


Figure 30: Lag compensator poles and zeros and its circuit implementation

The following figure shows frequency response simulation in MATLAB of lag compensator with various values of α.

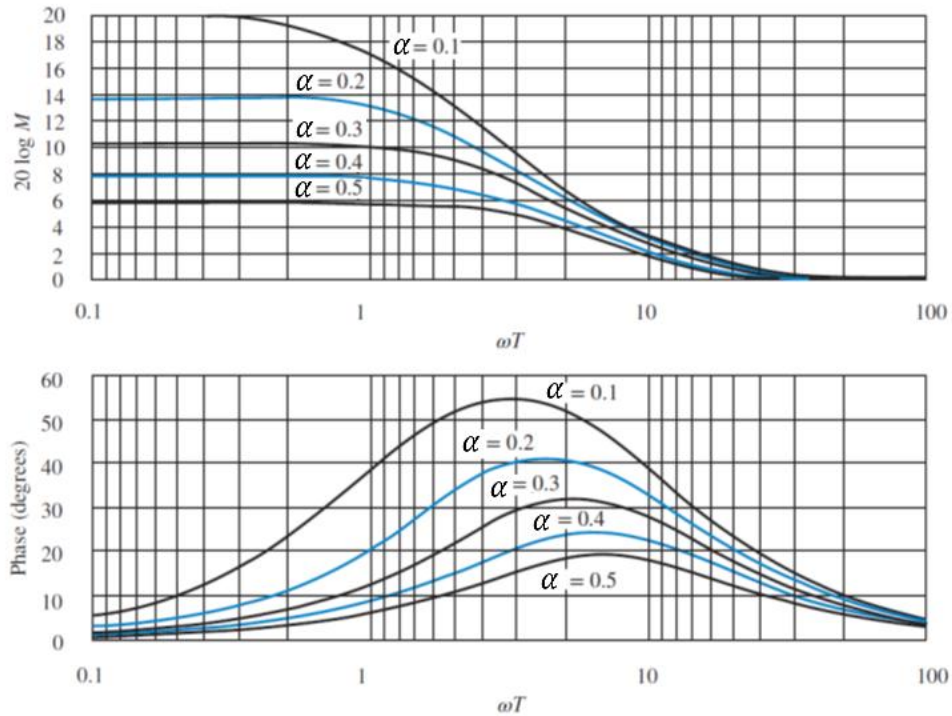


Figure 31: Frequency response of lag compensator with various values of α

3.2.2. Applications of Lag Compensators

In the lag compensator, the pole is closer to the origin than the zero, that is:

$$z_c > p_c$$

The lag compensator reduces steady-state error as shown in the figure below.

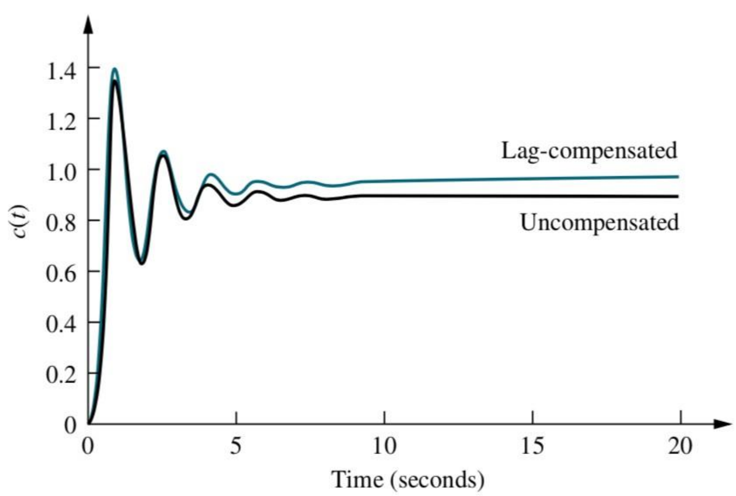


Figure 32: Transient responses of lag compensated and uncompensated system

3.3. Lead-Lag Compensator

For a lead-lag compensator as shown in the figure below, for its lead part, the zero is closer to the origin than the pole and for its lag part, the pole is closer to the origin than the zero.

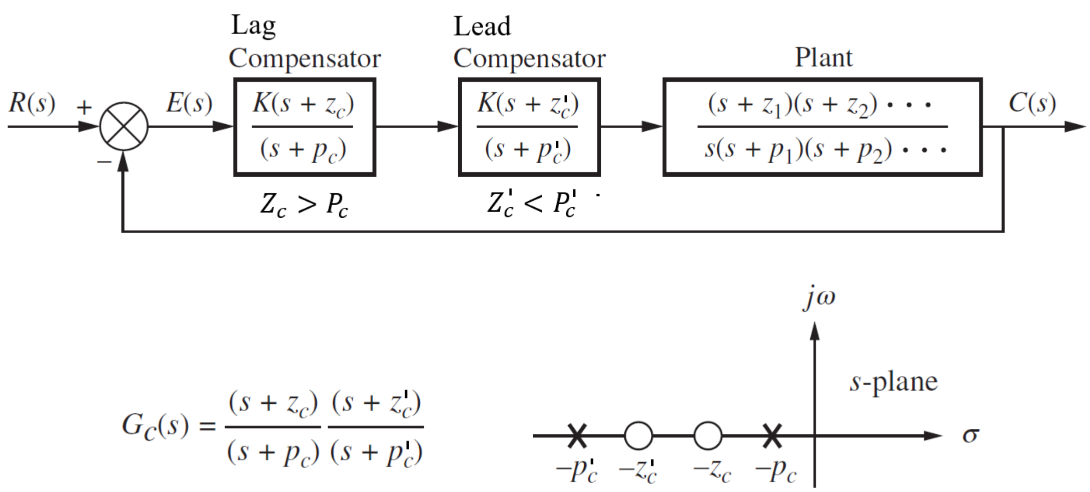


Figure 33: Lead-lag compensator in the control system

The transfer-function equation of a lead-lag compensator is:

$$G_c(s) = \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\gamma T_2}}\right) \qquad (\gamma > 1)$$

- Or

$$G_{lead-lag}(s) = G_{lead}(s)G_{lag}(s) = \left(\frac{s+z_{c(lag)}}{s+p_{c(lag)}}\right)\left(\frac{s+z_{c(lead)}}{s+p_{c(lead)}}\right)$$

Since $\alpha\beta = 1$, The pole and zero of the lag and lead parts of the lead-lag controller are:

$$|p_{c(lag)}| < |z_{c(lag)}| \quad \text{and} \quad |z_{c(lead)}| < |p_{c(lead)}|$$

3.3.1. Characteristics of Lead-Lag Compensators

Frequency response plot of lag compensator is as shown in the figure below.

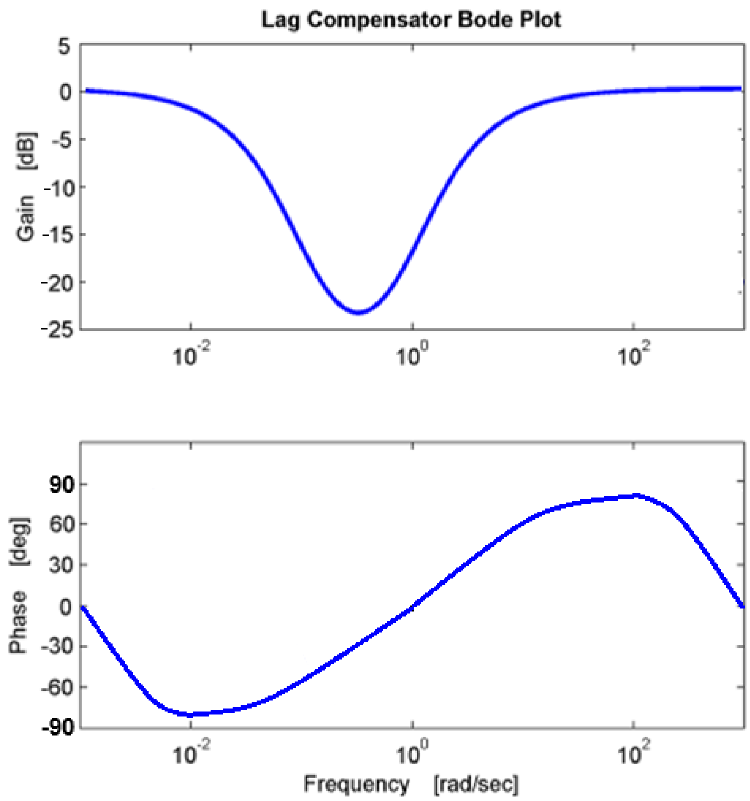


Figure 34: Frequency response of lead-lag compensator

- Magnitude plot:
  - Low: zero gain.
  - Cut-off 1: half gain.
  - Cut-off 2: half gain.
  - High: zero gain.
- Phase-shift plot:
  - Low: 0°.
  - Cut-off 1: -45°.
  - Cut-off 2: 45°.
  - High: 0°.

The lead-lag compensator can improve both steady-state error and transient response.

The lag pole of this compensator at $-p_{lag}$ and lag zero at $-z_{lag}$ are used to improve the steady-state error. Its lead pole at $-p_{lead}$ and lead zero at $-z_{lead}$ are used to improve the transient response. The lag pole at $-p_{lag}$ is small and negative and the lag zero at $-z_{lag}$ is close to, and to the left of, lag pole at $-p_{lag}$.

Its lead zero at $-z_{lead}$ and the lead pole at $-p_{lead}$ are selected to indicate the design point. Its lead pole at $-p_{lead}$ is more negative than lead zero at $-z_{lead}$.

In practice we do not need active circuits to implement it.

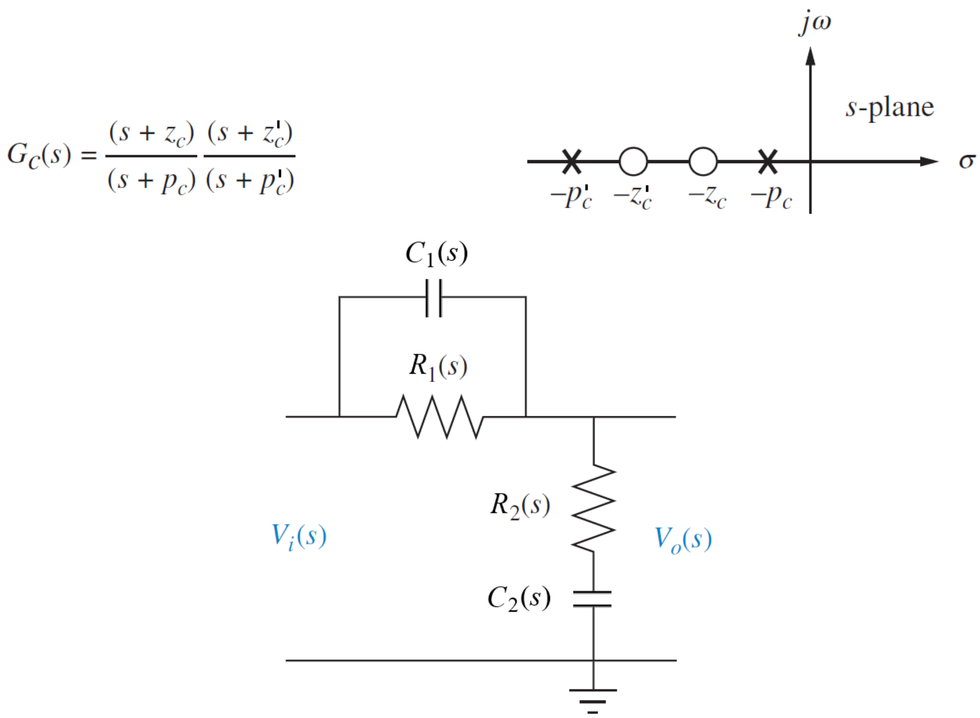


Figure 35: Lead-lag compensator poles and zeros and its circuit implementation

The figure given below shows frequency response simulation in MATLAB of lead-lag compensator with various values of γ are varied.

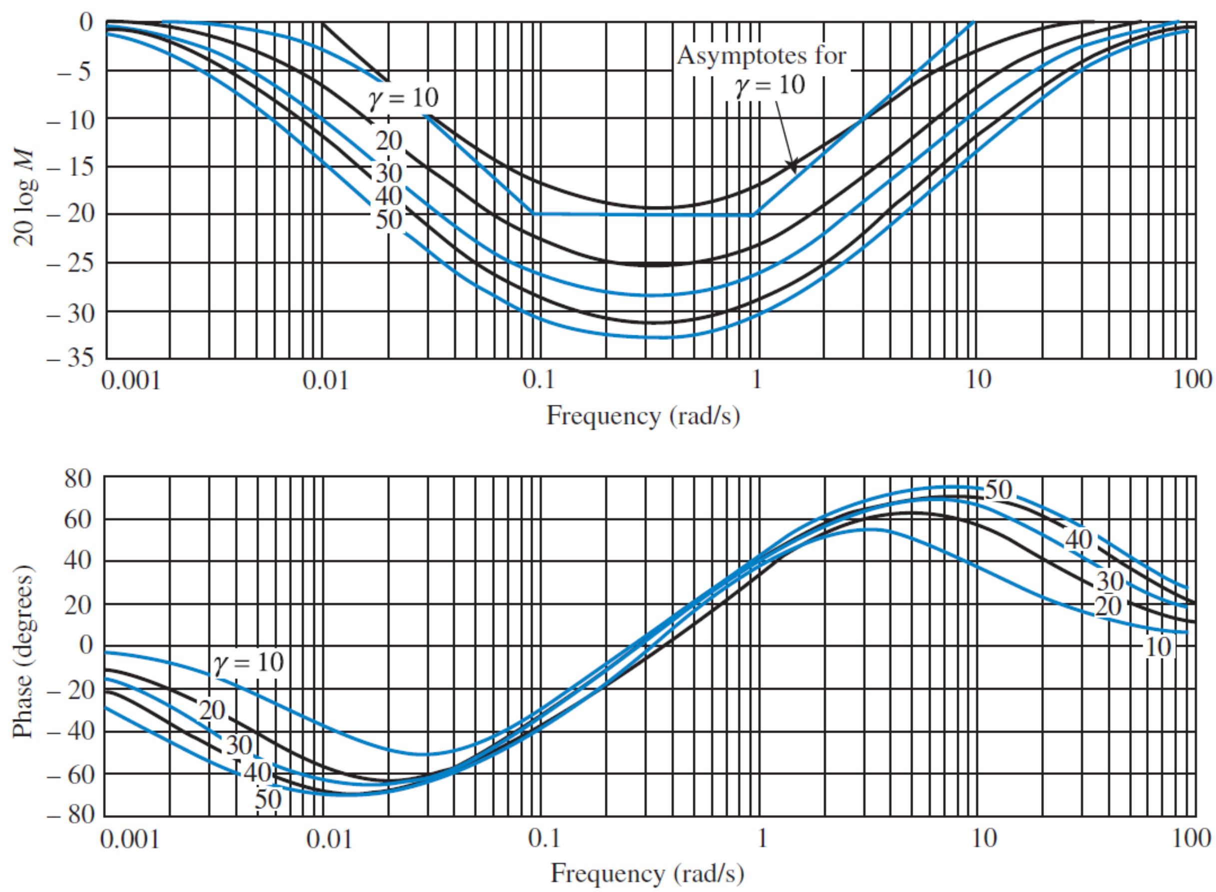


Figure 36: frequency response of lead-lag compensator with various values of γ

3.3.2. Applications of Lead-Lag Compensators

Considering the transfer-function equation of the second-order plant is:

$$G(s) = \frac{K}{(s+T_1)(s+T_2)}$$

For improving transient response, we can make z_c of the lag part in the $G_c(s)$ to be equal to the largest of T_1 and T_2, say T_2, to speed up the system.

Then, the pole in the lag part is used to cancel the zero of the lead-part of the compensator leaving pole of the lead part to be varied to meet the improvement goal or the design specification.

As a result, the transfer-function equation of the open-loop system becomes:

$$\frac{O(s)}{E(s)} = G_c(s)G(s) = \left(\frac{s+z_{c(lag)}}{s+p_{c(lag)}}\right)\left(\frac{s+z_{c(lead)}}{s+p_{c(lead)}}\right)\frac{K}{(s+T_1)(s+T_2)}$$

$$= \frac{K}{\left(s+p_{c(lead)}\right)(s+T_1)}$$

The transfer function equation of the closed-loop system is:

$$T(s) = \frac{O(s)}{I(s)} = \frac{\text{Forward}}{1 - \text{Loop}}$$

This gives:

$$T(s) = \frac{O(s)}{I(s)} = \frac{\left[\frac{K}{\left(s + p_{c(lead)}\right)(s + T_1)}\right]}{1 - \left(-\frac{K}{\left(s + p_{c(lead)}\right)(s + T_1)}\right)} = \frac{K}{\left(s + p_{c(lead)}\right)(s + T_1) + K}$$

$$= \frac{K}{s^2 + s\left(p_{c(lead)} + T_1\right) + \left(p_{c(lead)}T_1\right) + K}$$

The above transfer-function equation is a further example of pole-zero cancellation for system improvement. This time, we can have a complete control of the location of the pole for improving the transient response of the system.

Note: a pole is like $s + T_1$ term on the denominator and a zero is such a term on the numerator.

For improving the steady-state condition of the system, we can make z_c of the lead part in the $G_c(s)$ to be equal to the smaller of T_1 and T_2, say T_1, to remove more dominant pole in the system.

Then, the pole in the lead part is used to cancel the zero of the lag-part of the compensator leaving the pole of the lag part to be varied and assign to be very close to the origin e.g. $p_{c(lag)} \cong 0$ (i.e. to simulate an integral function like to the system). Thus, this would improve the steady-state condition of the system by reducing or removing the steady-state error in the system.

With this placement the system is able to meet its steady-state improvement goal or the design specification. As a result, the transfer-function equation of the open-loop system becomes:

$$\frac{O(s)}{E(s)} = G_c(s)G(s) = \left(\frac{s + z_{c(lag)}}{s + p_{c(lag)}}\right)\left(\frac{s + z_{c(lead)}}{s + p_{c(lead)}}\right)\frac{K}{(s + T_1)(s + T_2)} = \frac{K}{\left(s + p_{c(lag)}\right)(s + T_2)}$$

The transfer-function equation of the closed-loop system is:

$$T(s) = \frac{O(s)}{I(s)} = \frac{\text{Forward}}{1 - \text{Loop}}$$

This gives:

$$T(s) = \frac{O(s)}{I(s)} = \frac{\left[\frac{K}{\left(s + p_{c(lag)}\right)(s + T_2)}\right]}{1 - \left(-\frac{K}{\left(s + p_{c(lag)}\right)(s + T_2)}\right)} = \frac{K}{\left(s + p_{c(lag)}\right)(s + T_2) + K}$$

When $p_{c(lag)} \cong 0$, the equation above becomes:

$$T(s) = \frac{K}{s(s + T_2) + K} = \frac{K}{s^2 + sT_2 + K}$$

4. Intro to Compensation Design

We will focus on modifying system characteristics by applying feedback. Furthermore, we will be able to tailor the closed-loop transfer function with the addition of a compensator.

Compensator design is a compromise between two competing goals.

- Performance: Keeping the open loop gain high reduces system errors and the effects of disturbances.
- Stability: The closed-loop system must be kept stable by carefully managing the gain where the phase approaches −180°.

Compensator design can often be philosophically reduced to two (inter-related) problems e.g. one operating at low frequencies to achieve the required performance, the other at high frequency to ensure stability.

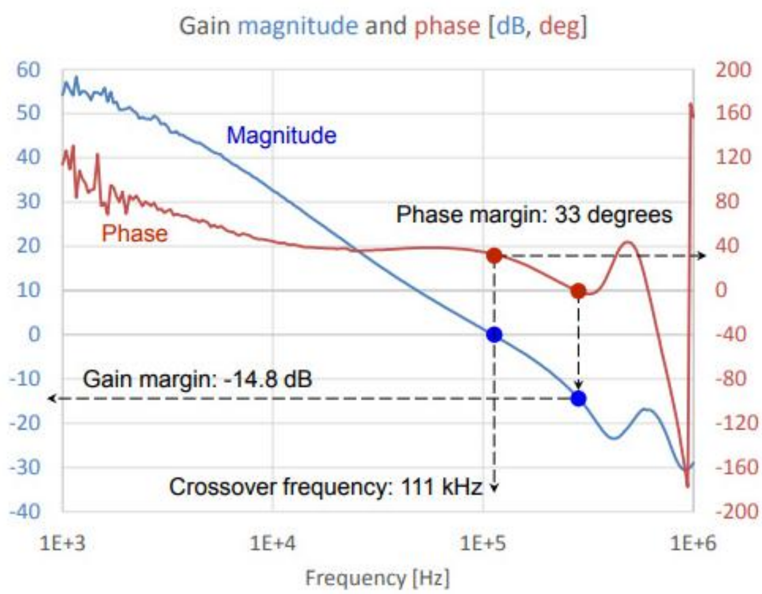


Figure 37: Frequency response of a given control system

4.1. Compensator Design

There are various approaches to designing a compensator:

1. Choose a compensator structure and then tune it manually.
2. Choose a compensator model and tune using a "recipe" (e.g. Ziegler-Nichols).

3. Use a model and solve for desired pole locations.
4. Measure the system performance and use a graphical technique.
5. Use a mathematical model with a graphical technique.
6. Use mathematical tools to achieve optimal performance (State-Space Analysis).

In the remaining lectures, we will focus on the graphical methods which form the classical control. These are mainly about items number 3 and 5 on the list.

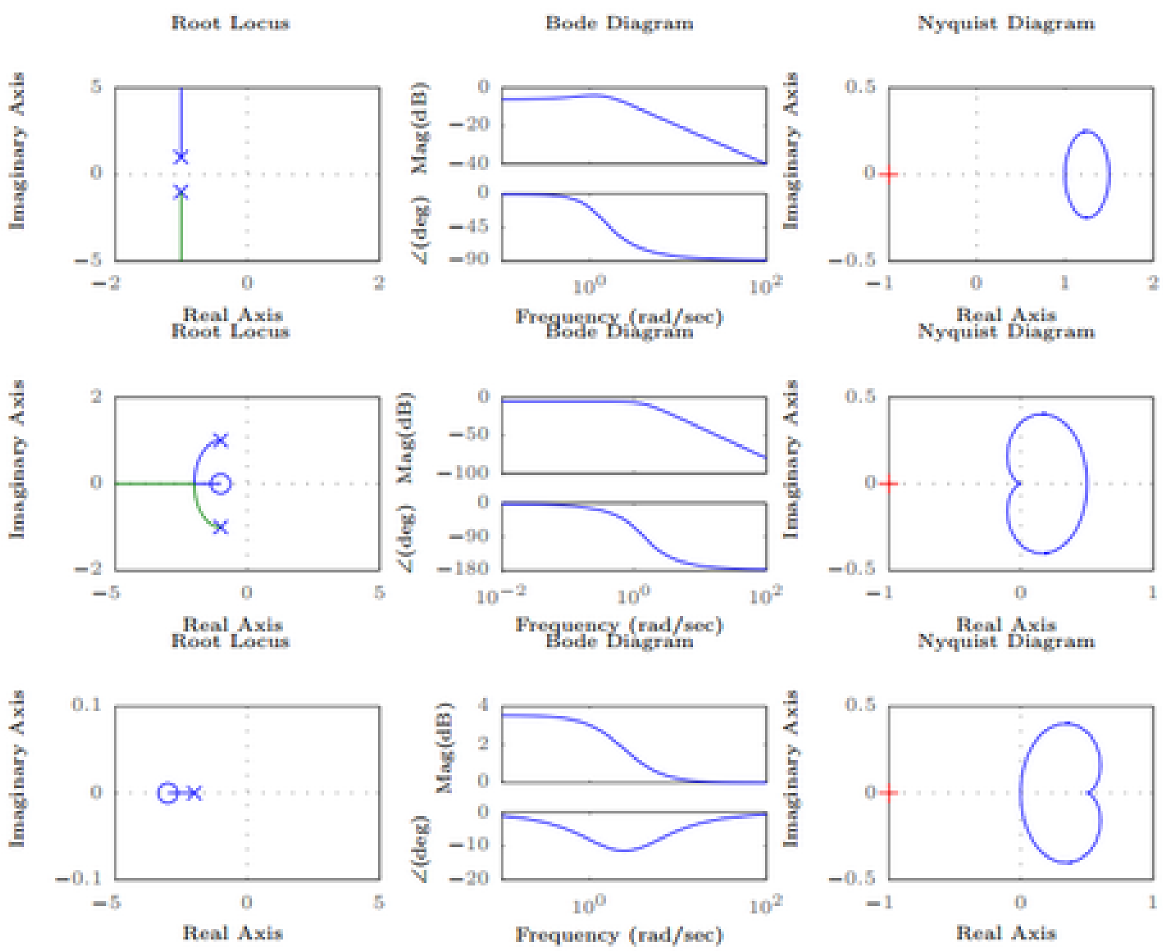


Figure 38: Root locus, Bode and Nyquist diagrams for controller/compensator design

4.2. System Topologies and Notations

We will generally design controller/compensators assuming unity gain feedback with the compensator $C(s)$ placed in the forward path.

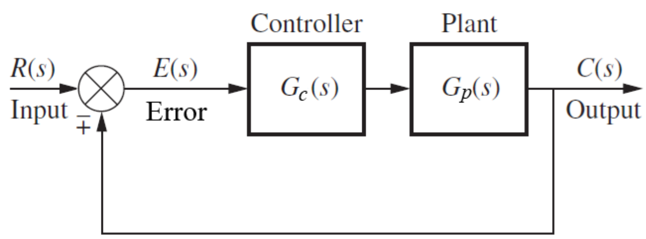


Figure 39: Typical placement of controller/compensator in the control systems

Remember that this is equivalent to a system with the controller/compensator in the feedback path.

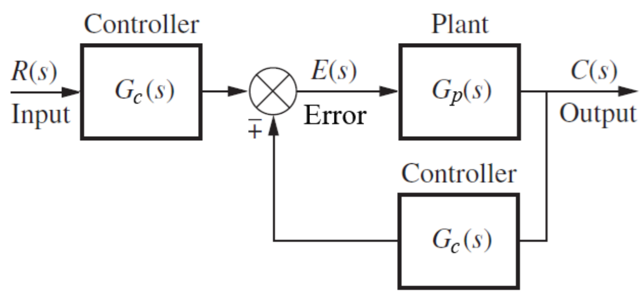


Figure 40: Equivalent arrangement of the controller/compensator in the control systems

Appendix 1: Standard Controllers and Compensators in Control Systems

Controller and Compensator	**Function**	**Transfer Function**	**Characteristics**
Controllers:			
P	Improve transient response (up to a point)	K	a. Increases the gain of the system. b. Often result in a non-zero steady-state error. c. Relatively easy to implement.
PI	Improve steady-state error	$K\left(\frac{s+z_c}{s}\right)$	a. Increases system type. b. Error becomes zero. c. Zero at z_c is small and negative. d. Active circuits are required to implement.
PD	Improve transient response	$K(s+z_c)$	a. Zero at $-z_c$ is selected to put the design point on the root locus. b. Active circuits are required to implement. c. It can cause noise and saturation; implement with rate feedback or with a pole (lead).
PID	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{s}\right]$	a. Lag zero at $-z_{lag}$ and pole at the origin improve steady-state error. b. Lag zero at $-z_{lag}$ is close to, and to the left of, the origin. c. Lead zero at $-z_{lead}$ improves transient response. d. Lead zero at $-z_{lead}$ is selected to put the design point on the root locus. e. Active circuits are required to implement. f. It can cause noise and saturation; implement with rate feedback or with an additional pole.
Compensators:			

Lag	Improve steady-state error	$\left(\frac{s+z_c}{s+p_c}\right)$ Where: $\lvert p_c\rvert < \lvert z_c\rvert$	a. Error is improved, but not driven to zero. b. Pole at $-p_c$ is small and negative. c. Zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. d. Active circuits are not required to implement.
Lead	Improve transient response	$\left(\frac{s+z_c}{s+p_c}\right)$ Where: $\lvert p_c\rvert > \lvert z_c\rvert$	a. Zero at $-z_c$ and pole at $-p_c$ at are selected to put design point on the root locus. b. Pole at $-p_c$ is more negative than zero at $-z_c$. c. Active circuits are not required to implement.
Lag-lead	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{(s+p_{lag})(s+p_{lead})}\right]$ Where: $\lvert p_{c(lag)}\rvert < \lvert z_{c(lag)}\rvert$ and $\lvert z_{c(lead)}\rvert < \lvert p_{c(lead)}\rvert$	a. Lag pole at $-p_{lag}$ and lag zero at $-z_{lag}$ are used to improve steady-state error. b. Lag pole at $-p_{lag}$ is small and negative. c. Lag zero at $-z_{lag}$ is close to, and to the left of, lag pole at $-p_{lag}$ d. Lead pole at $-p_{lead}$ and lead zero at $-z_{lead}$ are used to improve transient response. e. Lead zero at $-z_{lead}$ and the lead pole at $-p_{lead}$ are selected to put the design point on the root locus. f. Lead pole at $-p_{lead}$ is more negative than lead zero at $-z_{lead}$. g. Active circuits are not required to implement.