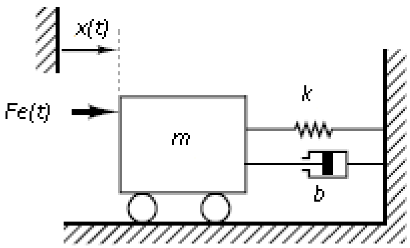
XMUT315 Control Systems Engineering

Final Exam Revision Questions (Solution)

A. System Modelling, Stability and Steady State

1. Given a mechanical system that consists of a mass that is separated from a wall by a spring and a damper. The mass (m) could represent a car, with the spring (spring coefficient, k) and damper (damper coefficient, b) representing the car's bumper. An external force ($F_e(t)$) is also shown. Only horizontal motion and forces are considered.



   a. Describe the state of equilibrium and the forces that are acting in the given system. [5 marks]

   b. Determine the transfer function of the system in terms of displacement ($x(t)$) over applied external force ($F_e(t)$). [10 marks]

   c. Describe models of system modelling. Why the approach used for modelling the system in this question is suitable for modelling the given system? [5 marks]

Solution

a. In the given mechanical system, there is only one position in this system defined by the variable "x" that is positive to the right. We assume that $x = 0$ when the spring is in its relaxed state. At the state of the equilibrium, we sum all the forces to zero.

   There are four forces which are acting in the given system:

   - An external force (F_e) applied on to the mass.
   - A force from the spring. To determine the direction, consider that the position "x" is defined positive to the right. If the mass moves in the positive "x" direction, the spring

is compressed and exerts a force on the mass. So, there will be a force from the spring ($f(\text{spring}) = kx$), to the left.

- A force from the damper. By an argument similar to that for the spring there will be a force from the damper ($f(\text{damper}) = bv$), to the left. The velocity, v, is the derivative of x with respect to time.

- Finally, there is the inertial force which is defined to be opposed to the defined direction of motion. This is represented by ($f(\text{mass}) = ma$) to the left. The acceleration, a, is the second derivative of x with respect to time. Do not forget this force as it is easy to do so.

b. The given mechanical system is modelled with a free body diagram as outlined as below:

Applying the Newton's second law, the transfer function of the system is calculated as follow, we sum all of these forces to zero at equilibrium and obtain:

$$\sum_{All\ forces} F = 0$$

From the free body diagram, the external force ($F_e(t)$) is opposed by all other forces existed in the system.

$$F_e(t) - ma(t) - bv(t) - kx(t) = 0$$

Or, normalize the equation above to functions of displacement ($x(t)$):

$$F_e(t) = m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t)$$

Applying Laplace transform to the equation above:

$$F_e(s) = ms^2x(s) + bs(xs) + kx(s)$$

Rearrange the equation given above, the transfer function of the system in terms of distance over external force is:

$$\frac{x(s)}{F_e(s)} = \frac{1}{ms^2 + bs + k}$$

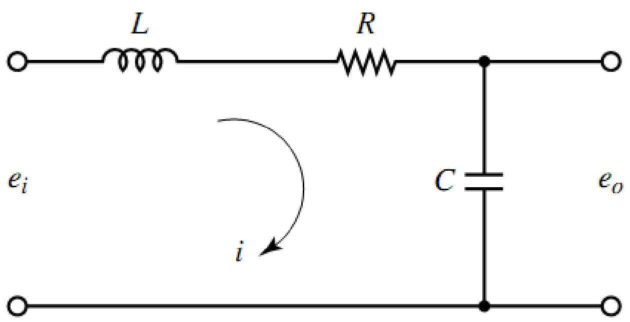
c. There are a number of approaches of modelling the physical systems:

- Scaled physical model: proportional to the actual model.
- Mathematical model: described as function and variable in mathematical equation.
- Numerical model: represented as a set of numbers to describe system characteristic and

behaviour.

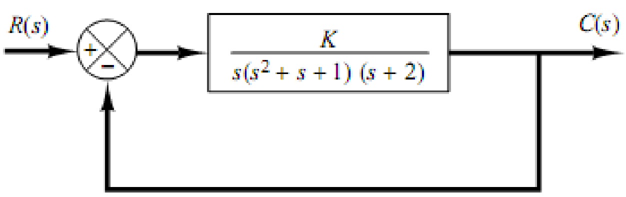
The mathematical model is suitable for this question as the mechanical system is sufficiently represented with the given modelling approach. Although forces that are acting on other directions such as friction, gravity and so forth are ignore in the given model, the salient characteristics of the given system can be shown with the respective free body diagram and mathematical equation.

2. Consider the electrical circuit shown in figure below.

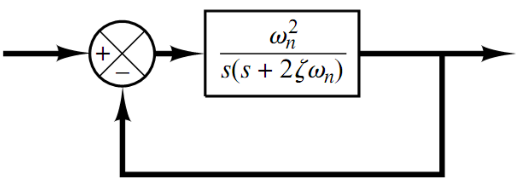


a. The circuit consists of an inductance L (Henry), a resistance R (Ohm), and a capacitance C (Farad). Determine the system transfer function. [10 marks]

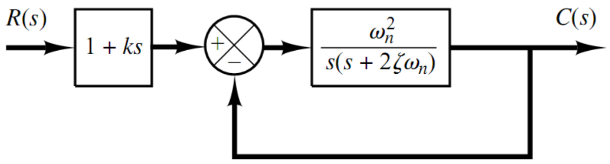
b. Determine the range of K for stability, when a controller is added to the circuit and the transfer function of the circuit becomes as shown below. [15 marks]



c. Consider the system shown in the figure given below. [5 marks]



The steady-state error to a unit-ramp input is $e(\infty) = 2\zeta\omega_n$. Show that the steady-state error for following a ramp input may be eliminated if the input is introduced to the system through a proportional-derivative controller, as shown in the figure below, and the value of k is properly set. Note that the error $e(t)$ is given by $r(t) - c(t)$.



Solution

a. Applying Kirchhoff's voltage law to the system, we obtain the following equations that give a mathematical model of the circuit.

For the input voltage

$$e_i = L\frac{di}{dt} + Ri + \frac{1}{C}\int i \ dt$$

And for the output voltage

$$e_o = \frac{1}{C}\int i\,dt$$

A transfer-function model of the circuit can also be obtained by taking the Laplace transforms of equations given above, assuming zero initial conditions, we obtain:

For the input voltage

$$E_i(s) = LsI(s) + RI(s) + \left(\frac{1}{C}\right)\left(\frac{1}{s}\right)I(s)$$

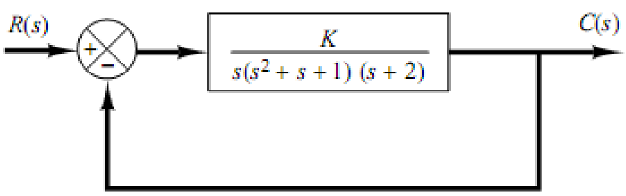
For the output voltage

$$E_o(s) = \left(\frac{1}{C}\right)\left(\frac{1}{s}\right)I(s)$$

If E_i is assumed to be the input and E_o the output, then the transfer function of this system is found to be:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

b. When a controller is added to the circuit, the circuit and controller becomes the system shown below.



The closed-loop transfer function of the system is given as:

$$\frac{C(S)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

For given system, the characteristic equation of the system is:

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Applying Routh-Hurwitz criteria, the array of coefficients becomes:

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	
s^1	$2-(9/7)K$		
s^0	K		

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$0 < K < \frac{14}{9}$$

When K = 14/9, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

Note that the ranges of design parameters that lead to stability may be determined by use of Routh-Hurwitz's stability criterion.

c. The closed-loop transfer function of the system shown in the figure above is:

$$\frac{C(s)}{R(s)} = \left(\frac{(1+ks)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

Then

$$R(s) - C(s) = \left(\frac{s^2+2\zeta\omega_n s-\omega_n^2 ks}{s^2+2\zeta\omega_n s+\omega_n^2}\right)R(s)$$

If the input ($R(s)$) is a unit ramp, then the steady-state error is:

$$e(\infty) = r(\infty) - c(\infty) = \lim_{s\to 0} s\left(\frac{s^2+2\zeta\omega_n s-\omega_n^2 ks}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\left(\frac{1}{s^2}\right) = \frac{2\zeta\omega_n-\omega_n^2 k}{\omega_n^2}$$

Therefore, if k is chosen as:

$$k = \frac{2\zeta}{\omega_n}$$

Then, the steady-state error for following a ramp input can be made equal to zero.

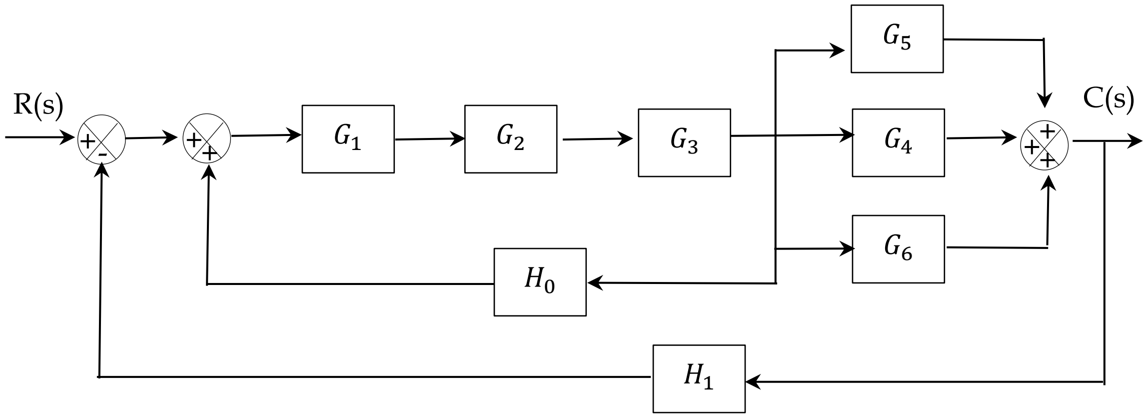
Note that, if there are any variations in the values of ζ and/or ω_n due to environmental changes or aging, then a nonzero steady-state error for a ramp response may result.

B. Block Diagram Manipulations, Feedback Control Systems, and Time Responses

3. Given a block diagram of a given feedback control system as shown in the figure below.

   a. Describe the guideline for simplifying block diagram. [5 marks]

   b. Simplify the block diagram given in the figure below to a single block. [15 marks]



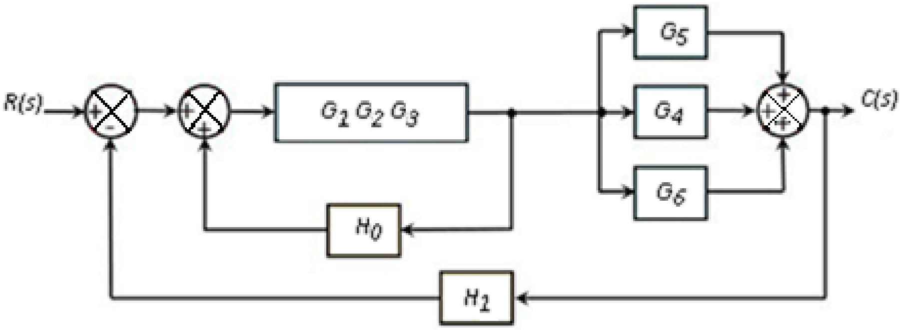
Solution

a. These are procedural steps to be followed for solving block diagram reduction problems:

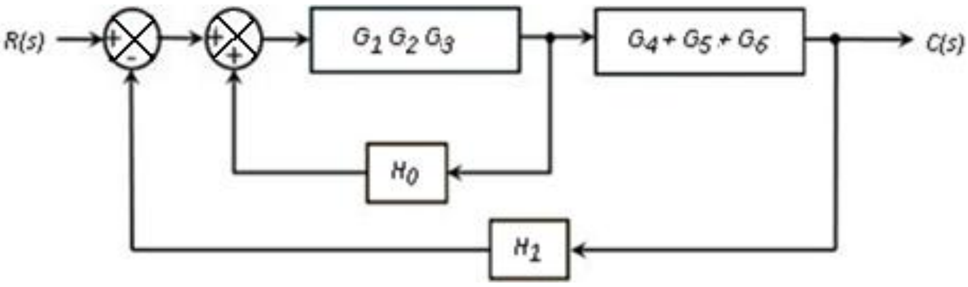
- The directly connected blocks in series must be reduced to a single block.
- Further, reduce the parallel-connected block into a single block.
- Now reduce the internally connected minor feedback loops.
- If shifting does not increase the complexity, then try having the take-off point towards the right while summing point towards left.
- Repeat the above-discussed steps to have a simplified system.
- Now determine the transfer function of the overall closed-loop simplified system.

b. The block diagram is simplified following the guideline as follows:

   Reducing the three blocks directly connected blocks in series (e.g. G_1, G_2, and G_3) into a single block, we will have:



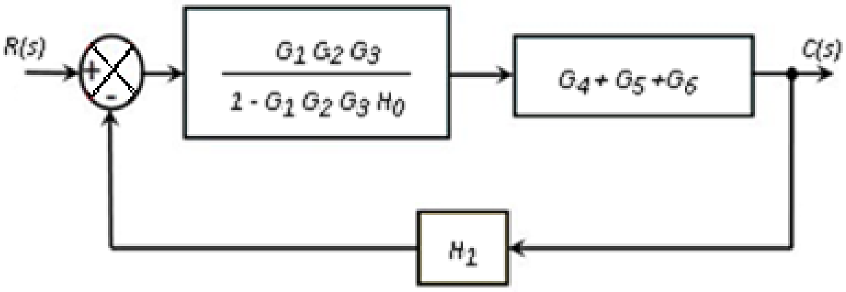
Further, we can see the three blocks (e.g. G_4, G_5, and G_6) are present that are connected in parallel. Thus, on reducing blocks in parallel, we will have:



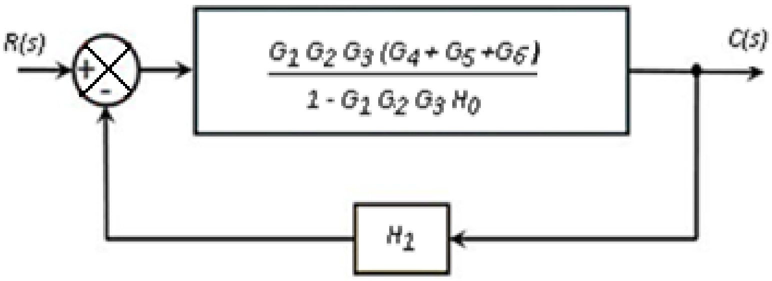
Further on simplifying the internal closed-loop system, the overall internal gain will be

$$\frac{G_1 G_2 G_3}{1 - G_1 G_2 G_3 H_0}$$

So, we will have:



Now, combining the two blocks in series:



So, this is the reduced canonical form of a closed-loop system. We know gain of the closed-loop system is given as:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore,

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3(G_4 + G_5 + G_6)}{1 - G_1G_2G_3H_0}}{1 + \left[\dfrac{G_1G_2G_3(G_4 + G_5 + G_6)}{1 - G_1G_2G_3H_0}\right]H_1}$$

On simplifying the equation, the transfer function of the block diagram becomes:

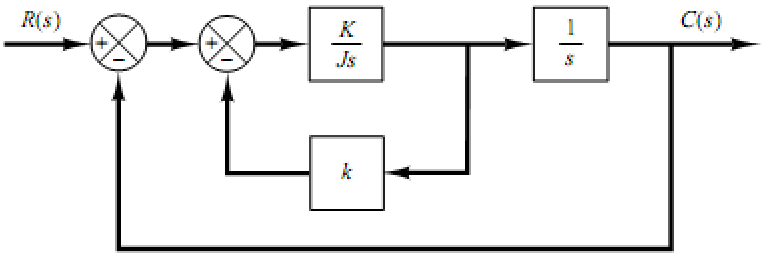
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3(G_4 + G_5 + G_6)}{1 - G_1G_2G_3H_0 + G_1G_2G_3(G_4 + G_5 + G_6)H_1}$$

4. You are provided with a feedback control system as given in the figure below.

   a. Perform block diagram reduction of the given system. [5 marks]

   b. Consider the system, where ζ = 0.6 and ω_n = 5 rad/sec, obtain the rise time T_r , peak time T_p, maximum overshoot M_p, and settling time T_s when the system is subjected to a unit-step input. [15 marks]

   c. Determine the values of K and k of the closed-loop system, so that the maximum overshoot in unit-step response is 25 % and the peak time is 2 sec. Assume that J = 1 kg-m^2. [10 marks]



Solution

a. The feedback control system becomes as shown below:

$$F(s) = \frac{G(s)}{1 + G(s)H(s)}$$

Simplify the inner feedback loop:

$$F_{inner}(s) = \frac{\left(\frac{K}{Js}\right)}{1 + \left(\frac{K}{Js}k\right)} = \frac{K}{Js + kK}$$

Simplify the outer feedback loop

$$F_{outer} = \frac{\left(\frac{K}{Js + kK} \cdot \frac{1}{s}\right)}{1 + \left(\frac{K}{Js + kK} \cdot \frac{1}{s}\right)} = \frac{K}{s(Js + kK) + K}$$

b. From the given values of ζ and ω_n, we obtain:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 5 \times \sqrt{1 - (0.6)^2} = 4$$

And

$$\sigma = \zeta\omega_n = 0.6 \times 5 = 3$$

Rise Time (T_r):

The rise time is:

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

Where β is given by:

$$\beta = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 0.93 \text{ rad}$$

The rise time (T_r) is thus:

$$T_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak Time (T_p):

The peak time is:

$$T_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum Overshoot (M_p):

The maximum overshoot is:

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling Time (T_s):

For the 2% criterion, the settling time is:

$$T_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion, the settling time is:

$$T_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

c. The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting J = 1 kg-m^2 into this last equation, we have:

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem $\omega_n = \sqrt{K}$ and $2\zeta\omega_n = Kk$, the maximum overshoot M_p is:

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Which is specified as 25 %. Hence

$$e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 0.25$$

From which:

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386 \qquad \text{or} \qquad \zeta = 0.404$$

The peak time T_p is specified as 2 sec and so:

$$T_p = \frac{\pi}{\omega_d} = 2 \qquad \text{or} \qquad \omega_d = 1.57$$

Then, the undamped natural frequency ω_n is:

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{1.57}{\sqrt{1-(0.404)^2}} = 1.72$$

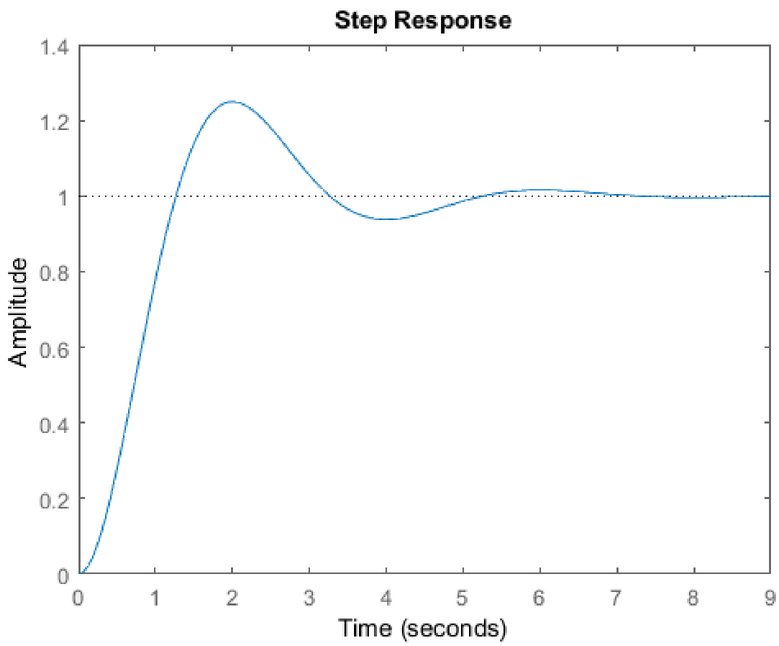
Therefore, we obtain:

$$K = \omega_n^2 = (1.72)^2 = 2.95 \text{ N.m}$$

As a result, the value of k is:

$$k = \frac{2\zeta\omega_n}{K} = \frac{(2)(0.404)(1.72)}{2.95} = 0.471$$

From MATLAB simulation, the transient response of the closed loop system is given in the graph below.



From the graph, the percentage overshoot of the system after given a unit-step is 25 % and the peak time is 2 secs. This confirms the validity of the results obtained from the calculations.

C. Controller/Compensator

5. Describe which controller or compensator will be able to fix the following problems in control system.

    a. Unstable system. [5 marks]

    b. Sluggish (slow) system. [5 marks]

    c. Large steady-state error. [5 marks]

    d. Both transient response and stability problems. [5 marks]

    e. Large steady-state error, but we wish to preserve the transient response characteristics. [5 marks]

Solution

a. For unstable system, PD controller or lead compensator would introduce additional gain or phase margins to the given system.

b. For sluggish (slow) system could be improved by introducing a proportional compensator/controller or a PD controller. This compensator or controller would make the system more responsive by reducing the damping of the system.

c. PI controller or lag compensator would be able to fix the system that experiences steady-state error. The controller or compensator introduces an integral function to the system that would eliminate the steady-state error of the system.

d. System with both transient and stability problems could be improved through application of lead-lag compensator or PID controller. These compensator and controller fix the transient and stability as they have both the properties of lag and lead compensators or PI and PD controllers.

e. For system with steady-state error, but we wish to preserve its transient response characteristics, we could apply lag compensator as this introduce gain at low frequency, but not at high frequency. As a result, the gain of the system at low frequency is improved, hence improvement in the steady-state error of the system, but the gain at higher frequency is not affected or changing.

6. Describe briefly the dynamic characteristics of the PI controller, PD controller, and PID controller. Compare these controllers with their respective counterparts e.g. lag, lead, and lag-lead compensators. [15 marks]

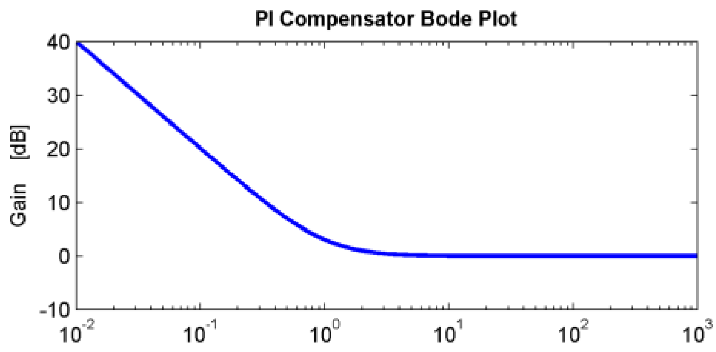
Solution

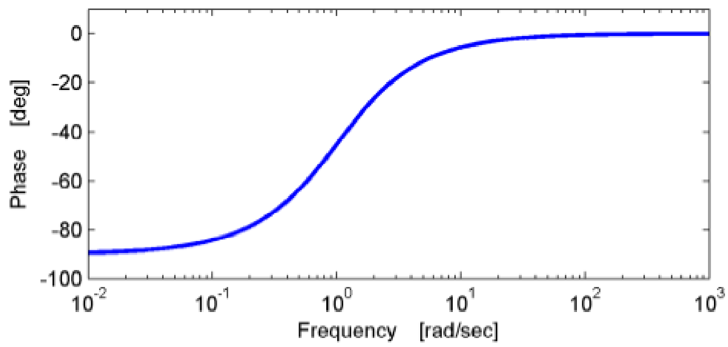
PI Controller:

The PI controller is characterized by the transfer function:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

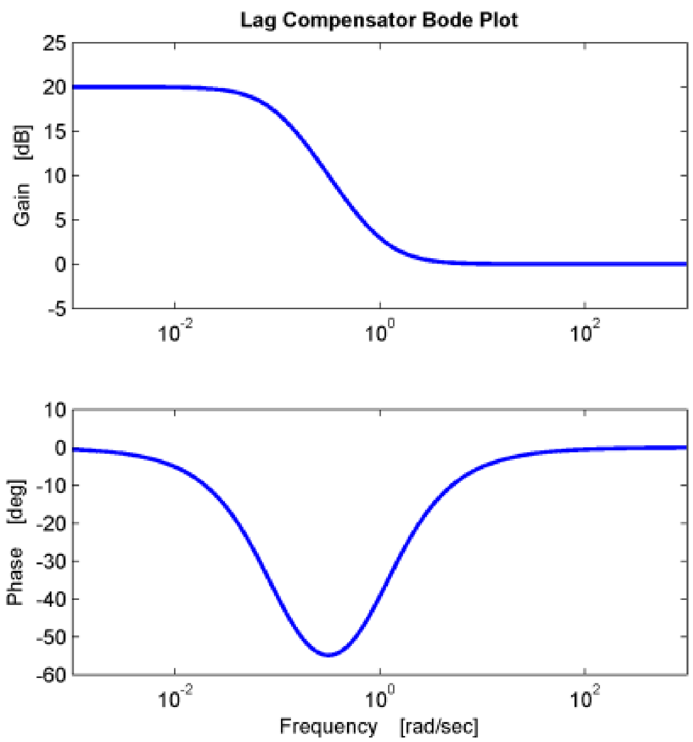
The PI controller is a variant of the lag compensator. It possesses a zero at $s = -1/T_i$ and a pole at $s = 0$. Thus, the characteristic of the PI controller is infinite gain at zero frequency. This improves the steady-state characteristics.





However, inclusion of the PI control action in the system increases the type's number of the compensated system by 1, and this causes the compensated system to be less stable or even makes the system unstable.

Therefore, the values of must be chosen carefully to ensure a proper transient response. By properly designing the PI controller, it is possible to make the transient response to a step input exhibit relatively small or no overshoot. The speed of response, however, becomes much slower. This is because the PI controller, being a low-pass filter, attenuates the high-frequency components of the signal.



PD Controller:

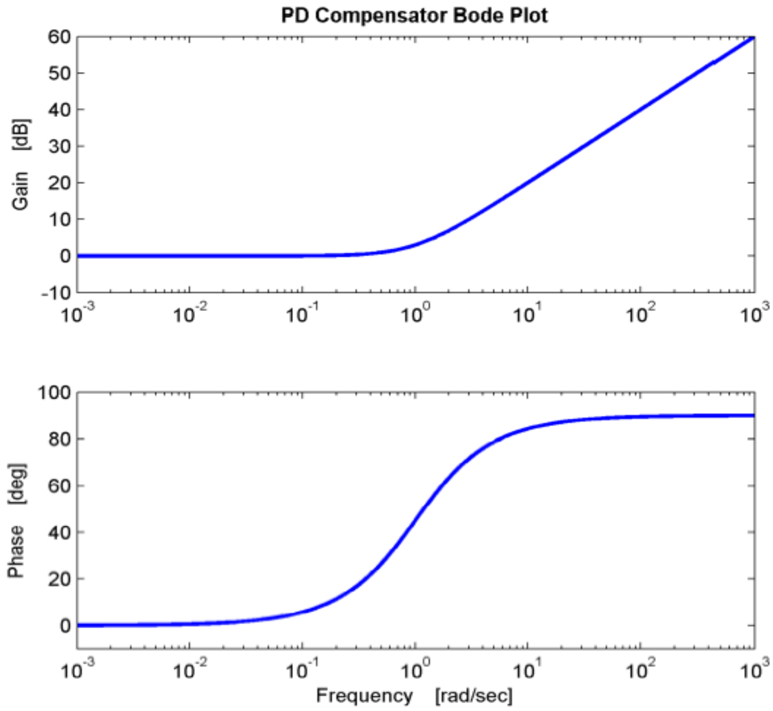
The PD controller is a simplified version of the lead compensator. The PD controller has the transfer function where:

$$G_c(s) = K_p(1 + T_d s)$$

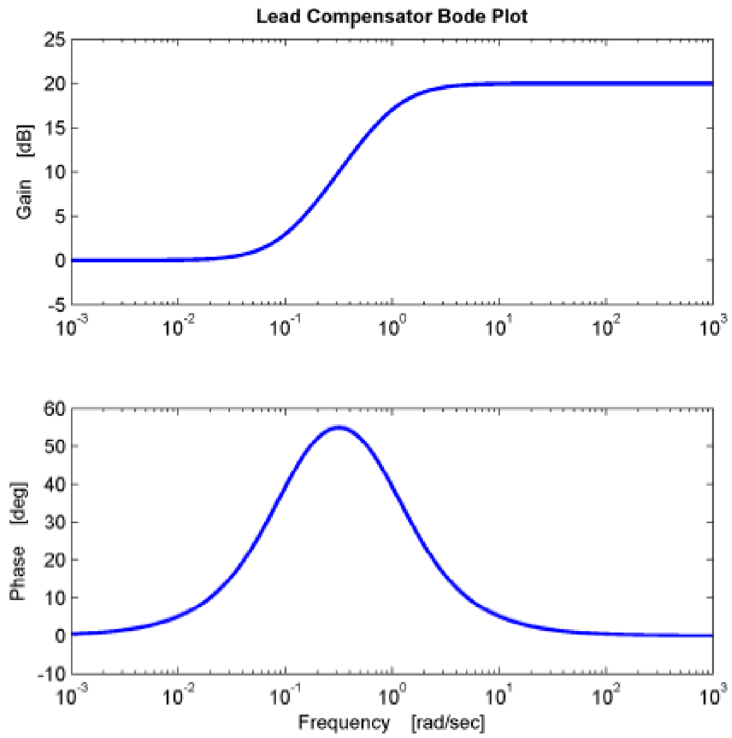
The value of K_p is usually determined to satisfy the steady-state requirement. The corner frequency is chosen such that the phase lead occurs at around the gain crossover frequency.

Although the phase margin can be increased, the magnitude of the compensator continues to increase for the frequency region (Thus, the PD controller is a high-pass filter.) Such a continued increase of the magnitude is undesirable, since it amplifies high-frequency noises that may be present in the system.

Lead compensation can provide a sufficient phase lead, while the increase of the magnitude for the high-frequency region is very much smaller than that for PD control. Therefore, lead compensation is preferred over PD control.



Because the transfer function of the PD controller involves one zero, but no pole, it is not possible to electrically realize it by passive RLC elements only. Realisation of the PD controller using op amps, resistors, and capacitors is possible, but because the PD controller is a high-pass filter, as mentioned earlier, the differentiation process involved may cause serious noise problems in some cases.



There is, however, no problem if the PD controller is realised by use of the hydraulic or pneumatic elements. The PD control, as in the case of the lead compensator, improves the transient-response characteristics, improves system stability, and increases the system bandwidth, which implies fast rise time.

PID Controller:

The transfer function of a PID controller is given as:

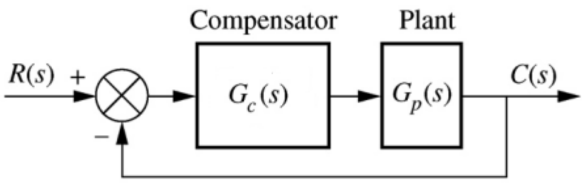
$$K\left[\frac{(s+z_{lag})(s+z_{lead})}{s}\right]$$

The PID controller is a combination of the PI and PD controllers. It is a lag–lead compensator. Note that the PI control action and PD control action occur in different frequency regions. The PI control action occurs at the low-frequency region and PD control action occurs at the high-frequency region. The PID control may be used when the system requires improvements in both transient and steady-state performances.

PID controller is very similar to lag-lead compensator. Both controllers/compensators are used for improving both transient response and steady-state condition of the control system. Both have integration and differentiation functions in their transfer functions.

But, there are also their differences as in a number of cases compensator is preferred rather than controller for solving the control system problems due to its simplicity in its implementation. Compensator also can be created with passive components, not active components like the controller. Controller is preferable if you require some sorts of control flexibility in its application compared with the compensator.

7. Consider a unity-feedback control system as given below with a compensator arranged in series with the plant.



Given the compensator ($G_c(s)$) is a proportional controller with gain K and the transfer function of the plant is given as:

$$G_P(s) = \frac{1}{s(s+3)}$$

Perform the following tasks:

a. For the steady-state error analysis, determine the steady-state error of the system to unit step and unit ramp inputs. Describe how you can improve the steady-state error of the system. [10 marks]

b. For the transient response analysis, derive equation for damping ratio of the system. If we require a damping ratio of 0.5, determine the gain of the proportional controller. [10 marks]

Solution

a. For the steady-state error analysis, evaluate the loop transfer function of the system:

$$G(s) = G_c(s)G_p(s) = K\frac{1}{s(s+3)} = \frac{K}{s(s+3)}$$

So, the system above is Type 1, as a result, its steady-state error $e(\infty)$ is 0 for a step input $r(t) = u(t)$.

But, for a unit ramp input, the steady-state error of the system is:

$$e(\infty) = \frac{1}{K_v}$$

The value of the velocity-error constant is:

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left[\frac{K}{s(s+3)}\right] = \frac{K}{3}$$

Thus, the steady-state error ($e(\infty)$) when the system is given a unit ramp input is:

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{3}{K}$$

To obtain small steady-state error to unit ramp input you require large gain K.

b. For transient response analysis, the closed-loop transfer function of the system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+3)}}{1+\frac{K}{s(s+3)}} = \frac{K}{s(s+3)+K} = \frac{K}{s^2+3s+K}$$

Comparing with standard second-order system's case:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$$

So

$$\omega_n = \sqrt{K} \qquad \text{and} \qquad 2\omega_n\zeta = 3$$

Thus, the damping ratio of the system is:

$$\zeta = \frac{3}{2\sqrt{K}}$$

The gain of the proportional controller (K) is calculated as follows:

$$\frac{3}{2\sqrt{K}} = 0.5$$

Rearrange the above equation the gain of the proportional controller when damping ratio of the system is 0.5 is:

$$K = (3)^2 = 9$$

D. Bode Plots

8. Consider a unity-feedback system whose open-loop transfer function is:

$$G(s) = \frac{(s+10)}{s(s+1)(s+6)}$$

a. Describe briefly the frequency response of the system. Calculate the gains, phase shifts and slopes at DC, low frequency, break points frequencies and high frequency. [15 marks]

b. Sketch the Bode plots (magnitude and phase) of the frequency response of the system. [10 marks]

c. Indicate in the plots the gain and phase margins of the system. Describe the stability of the system. [5 marks]

d. When the system is subjected to a ramp input, determine the velocity error constant (K_v) from the plots. Determine the value of this error constant from the plot. [10 marks]

e. Based on the results in part (d), how you improve the steady-state characteristics of the system when it is subjected to a ramp input. [5 marks]

Solution

a. Looking at the frequency response of the given control system, it has a gain at -20 dB/decade at 0 rad/s (DC) due to its pole at origin. It has several break points such as simple poles at 1 rad/s and 6 rad/s and a zero at 10 rad/s. The gain settles to -40 dB/decade slope at high frequency.

For the phase shift of the frequency response, its phase shift at -90° at 0 rad/s (DC) due to pole at origin and the system has several break points as described above, and the phase shift of the system settles down to -180° at high frequency.

The important points and slopes in frequency response of the system is calculated below.

Magnitude or gain of the system:

$$|G(j\omega)| = K\frac{\prod_{i=1}^{m}|Z_i|}{\prod_{i=1}^{m}|P_i|}$$

- DC gain: a gain of +∞.

$$|G(0)| = 20\log\left|\frac{(s+10)}{s(s+1)(s+6)}\right|_{\omega=0} = 20\log\left|\frac{\left(\sqrt{0^2+10^2}\right)}{0\left(\sqrt{0+1^2}\right)\left(\sqrt{0^2+6^2}\right)}\right| = +\infty\text{ dB}$$

- At frequency range between low frequency to breakpoint at 1 rad/s: a slope at -20 dB per rad/s.
- Gain at 1 rad/s:

$$|G(1)| = 20\log\left|\frac{(s+10)}{s(s+1)(s+6)}\right|_{\omega=1} = 20\log\left|\frac{\left(\sqrt{1^2+10^2}\right)}{1\left(\sqrt{1^2+1^2}\right)\left(\sqrt{1^2+6^2}\right)}\right| = 1.3\text{ dB}$$

- At frequency range between break point 1 rad/s to break point at 6 rad/s: a slope at -40 dB/decade.
- Gain at 6 rad/s:

$$|G(6)| = 20\log\left|\frac{(s+10)}{s(s+1)(s+6)}\right|_{\omega=6} = 20\log\left|\frac{\left(\sqrt{6^2+10^2}\right)}{6\left(\sqrt{6^2+1^2}\right)\left(\sqrt{6^2+6^2}\right)}\right| = -28.48\text{ dB}$$

- At frequency range between break point at 6 rad/s and 10 rad/s: a slope at -60 dB/decade.
- Gain at 10 rad/s:

$$|G(10)| = 20\log\left|\frac{(s+10)}{s(s+1)(s+6)}\right|_{\omega=10}$$

$$= 20\log\left|\frac{\left(\sqrt{10^2+10^2}\right)}{10\left(\sqrt{10^2+1^2}\right)\left(\sqrt{10^2+6^2}\right)}\right| = -38.37\text{ dB}$$

- At frequency range between break point at 10 rad/s and high frequency: a slope at -40 dB/decade.

Phase shift of the system:

$$\angle G(j\omega) = \sum_{i=1}^{m}\angle Z_i - \sum_{i=1}^{m}\angle P_i$$

- DC and low frequency:

$$\angle G(0) = \frac{\angle(s+10)}{\angle s + \angle(s+1) + \angle(s+6)} = \frac{\tan^{-1}\left(\frac{0}{10}\right)}{\tan^{-1}\left(\frac{0}{0}\right) + \tan^{-1}\left(\frac{0}{1}\right) + \tan^{-1}\left(\frac{0}{6}\right)} = -90°$$

- At phase range between low frequency and breakpoint at 1 rad/s: a slope at 45°/decade.
- Phase at breakpoint at 1 rad/s:

$$\angle G(1) = \frac{\angle(s+10)}{\angle s + \angle(s+1) + \angle(s+6)} = \frac{\tan^{-1}\left(\frac{1}{10}\right)}{\tan^{-1}\left(\frac{1}{0}\right) + \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{6}\right)}$$

$$= \frac{5.71°}{90° + 45° + 9.46°} = -138.75°$$

- At phase range between breakpoints at 1 rad/s and 6 rad/s: -90°/decade.
- Phase at breakpoint at 6 rad/s:

$$\angle G(6) = \frac{\angle(s+10)}{\angle s + \angle(s+1) + \angle(s+6)} = \frac{\tan^{-1}\left(\frac{6}{10}\right)}{\tan^{-1}\left(\frac{6}{0}\right) + \tan^{-1}\left(\frac{6}{1}\right) + \tan^{-1}\left(\frac{6}{6}\right)}$$

$$= \frac{30.96°}{90° + 80.54° + 45°} = -184.58°$$

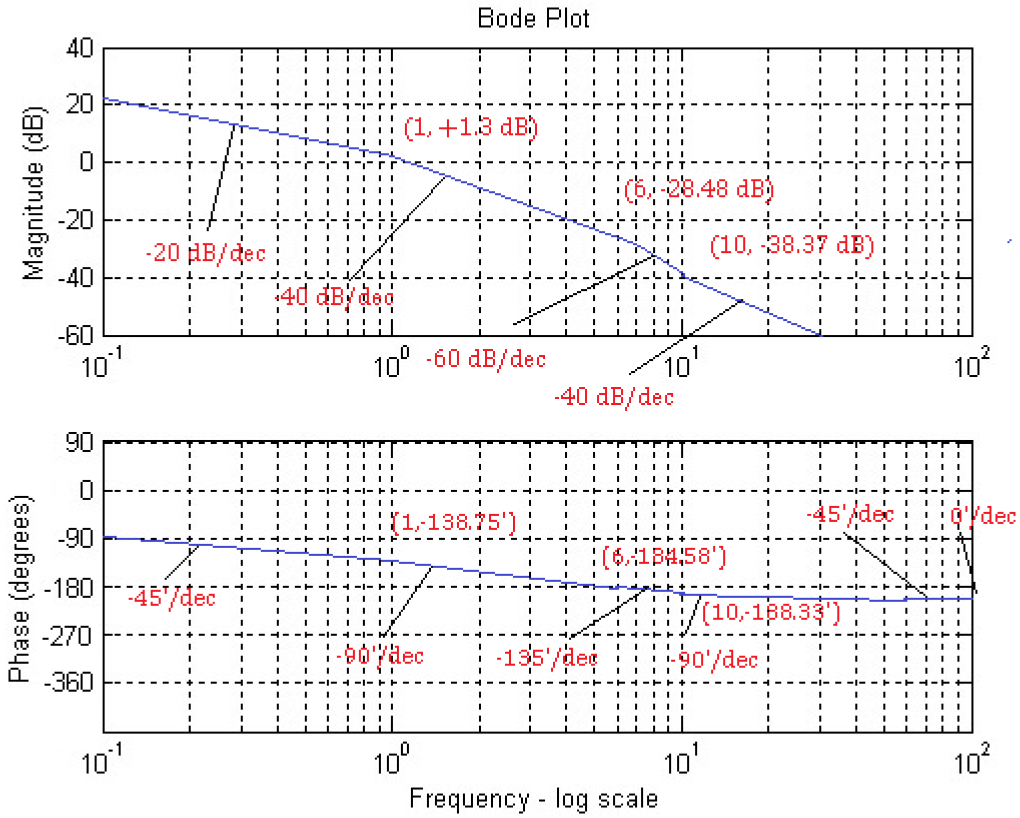
- At phase range between breakpoints at 6 rad/s and 10 rad/s: -135°/decade.
- Phase at breakpoint at 10 rad/s:

$$\angle G(10) = \frac{\angle(s+10)}{\angle s + \angle(s+1) + \angle(s+6)} = \frac{\tan^{-1}\left(\frac{10}{10}\right)}{\tan^{-1}\left(\frac{10}{0}\right) + \tan^{-1}\left(\frac{10}{1}\right) + \tan^{-1}\left(\frac{10}{6}\right)}$$

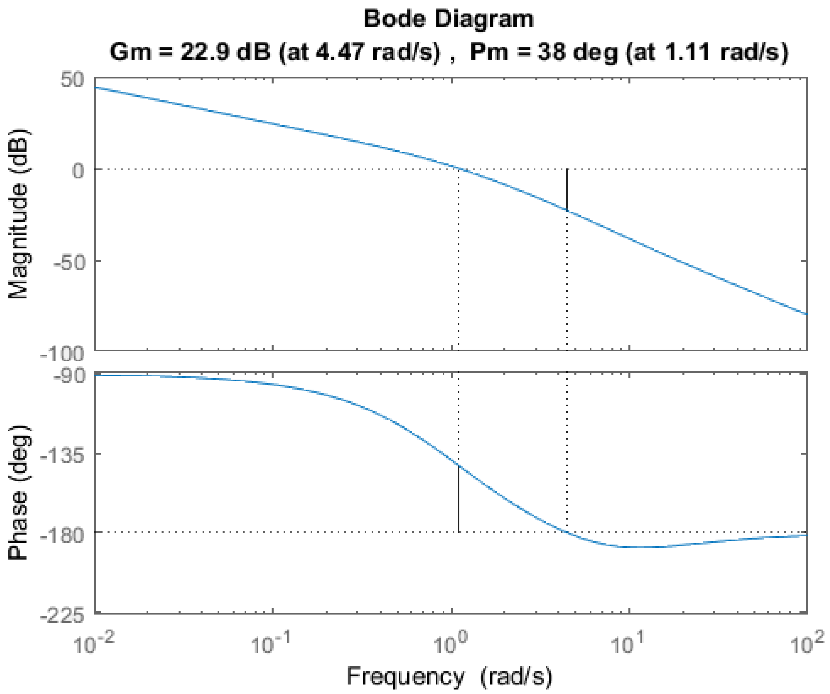
$$= \frac{45°}{90° + 84.29° + 59.04°} = -188.33°$$

- At phase range between breakpoints at 10 rad/s and high frequency: -90°/decade
- Higher frequency: Phase shift is approaching -180°

b. Sketch the Bode plots (magnitude and phase) of the frequency response of the system are shown in the figures below.



Note that the Bode plots of the system using MATLAB simulation are shown below.



c. The velocity error constant is determined in the plot as the gain at the intersection between the low frequency gain line extended to a vertical line at 0 dB.

When the system is subjected to a ramp input, the velocity error constant (K_v) is around about 5 dB (1.78) from the magnitude or gain plot.

Analytically, velocity error constant ((K_v)is found from:

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\left[\frac{(s+10)}{s(s+1)(s+6)}\right] = \frac{10}{(1)(6)} = 1.66$$

d. Since the system is a type 1, the steady-state error of the system when given a step input is zero. Analytically the static error constants of the system are as follows:

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0}\left[\frac{(s+10)}{s(s+1)(s+6)}\right] = \frac{10}{0(1)(6)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0}\left[s\frac{(s+10)}{s(s+1)(s+6)}\right] = \frac{10}{(1)(6)} = \frac{10}{6} = 1.66$$

$$K_a = \lim_{s \to 0} s^2G(s) = \lim_{s \to 0}\left[s^2\frac{(s+10)}{s(s+1)(s+6)}\right] = \frac{(0)(10)}{(1)(6)} = 0$$

The steady-state error of the system according to the input signals:

$$e_{step}(\infty) = \frac{1}{1+K_p} = 0$$

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{1}{(10/6)} = 0.6$$

$$e_{parabola}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

e. The steady-state characteristics of the system when it is subjected to a ramp input could be improved through the application of PI controller. With the integral function introduced by this type of controller, the steady-state error of the system when it is subjected to ramp input becomes zero.

Although it would not completely eliminate the steady-state error, if it is added into the given control system, the Lag compensator could minimise the steady-state error of the system.

9. You are given a control system with the following open-loop transfer function:

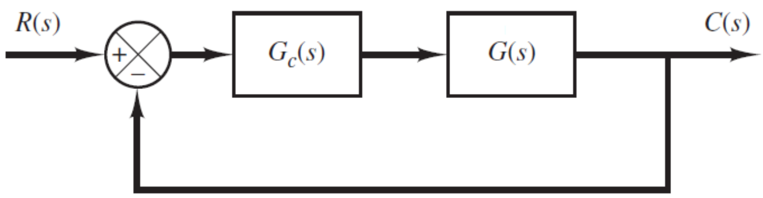
$$G(s) = \frac{1}{s(s+1)(2s+1)}$$

a. After determining the important points and slopes, sketch the frequency response of the control system using Bode plots (magnitude and phase). [10 marks]

b. From the graph in part (a), determine the gain margin (GM) and phase margin (PM) of the system. [5 marks]

c. Based on the gain and phase margins of the system, describe the stability of the system. [5 marks]

d. If the system is connected in series with a proportional controller ($G_c(s)$) with a gain of K as shown below, find the critical value of the gain K for stability. [15 marks]



e. Is a closed-loop system with $K = 2$ stable? How do you improve the stability of the system if it is unstable? [5 marks]

Solution

a. Sketches the Bode plots (magnitude and phase) of the system are calculated.

The points to be drawn in the sketches are determined as follows:

Magnitude or gain of the system:

$$|G(j\omega)| = K\frac{\prod_{i=1}^{m}|Z_i|}{\prod_{i=1}^{m}|P_i|}$$

- DC gain: a gain of +∞.

$$|G(0)| = 20\log\left|\frac{1}{s(s+1)(2s+1)}\right|_{\omega=0} = 20\log\left|\frac{1}{0\left(\sqrt{0+1^2}\right)\left(\sqrt{2(0)^2+1^2}\right)}\right| = +\infty \text{ dB}$$

- At frequency range between low frequency to breakpoint at 0.5 rad/s: a slope at -20 dB per rad/s.
- Gain at 0.5 rad/s:

$$|G(0.5)| = 20\log\left|\frac{1}{s(s+1)(2s+1)}\right|_{\omega=0.5}$$
$$= 20\log\left|\frac{1}{0.5\left(\sqrt{0.5^2+1^2}\right)\left(\sqrt{2(0.5)^2+1^2}\right)}\right| = +3.29 \text{ dB}$$

- At frequency range between break point 0.5 rad/s to break point at 1 rad/s: a slope at -40 dB/decade.
- Gain at 1 rad/s:

$$|G(1)| = 20\log\left|\frac{1}{s(s+1)(2s+1)}\right|_{\omega=1} = 20\log\left|\frac{1}{1\left(\sqrt{1^2+1^2}\right)\left(\sqrt{2(1)^2+1^2}\right)}\right| = -7.78 \text{ dB}$$

- At frequency range between break point at 1 rad/s and high frequency: a slope at -60 dB/decade.

Phase shift of the system:

$$\angle G(j\omega) = \sum_{i=1}^{m} \angle Z_i - \sum_{i=1}^{m} \angle P_i$$

- DC and low frequency:

$$\angle G(0) = \frac{1}{\angle s + \angle(s+1) + \angle(2s+1)} = \frac{1}{\tan^{-1}\left(\frac{0}{0}\right) + \tan^{-1}\left(\frac{0}{1}\right) + \tan^{-1}\left(\frac{0}{0.5}\right)} = -90°$$

- At phase range between low frequency and breakpoint at 0.5 rad/s: a slope at 45°/decade.
- Phase at breakpoint at 0.5 rad/s:

$$\angle G(0.5) = \frac{1}{\angle s + \angle(s+1) + \angle(2s+1)} = \frac{1}{\tan^{-1}\left(\frac{0.5}{0}\right) + \tan^{-1}\left(\frac{0.5}{1}\right) + \tan^{-1}\left(\frac{0.5}{0.5}\right)}$$
$$= \frac{1}{90° + 26.56° + 45°} = -161.56°$$

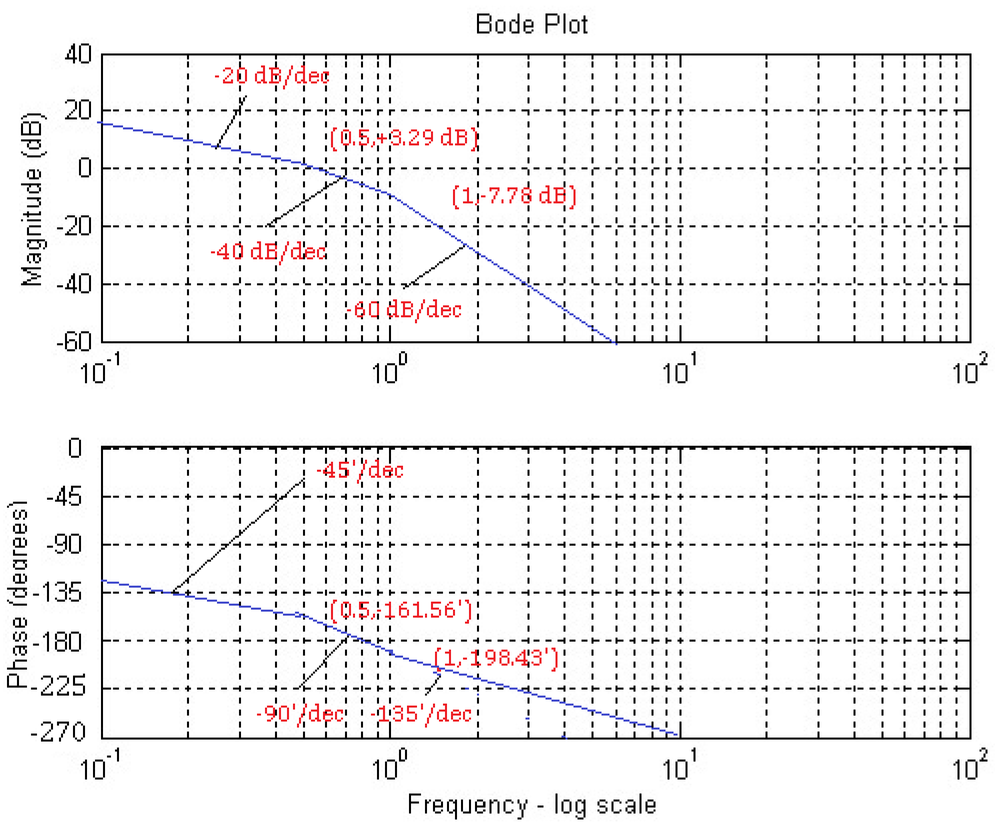
- At phase range between breakpoints at 0.5 rad/s and 1 rad/s: -90°/decade.
- Phase at breakpoint at 1 rad/s:

$$\angle G(6) = \frac{1}{\angle s + \angle(s+1) + \angle(2s+1)} = \frac{1}{\tan^{-1}\left(\frac{1}{0}\right) + \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{0.5}\right)}$$

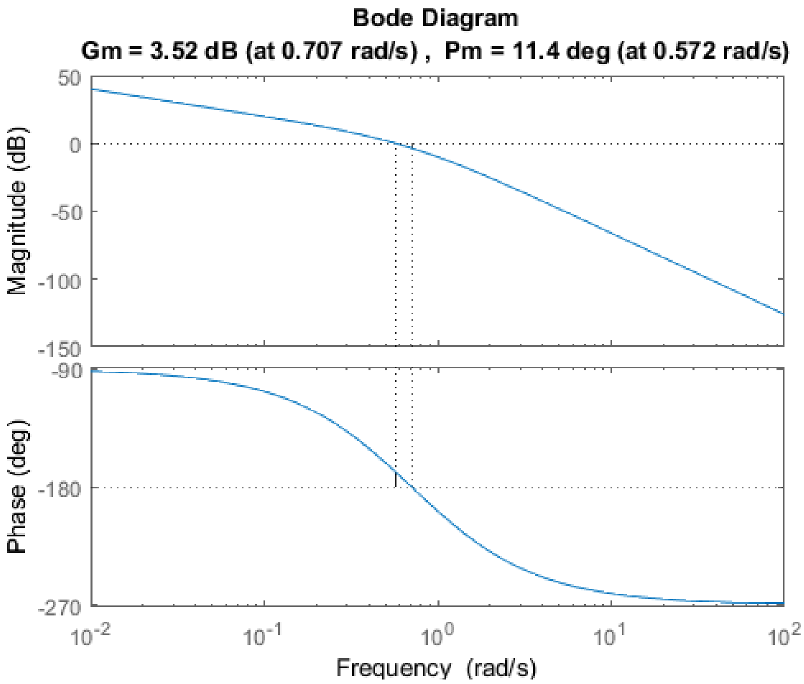
$$= \frac{1}{90° + 45° + 63.43°} = -198.43°$$

- At phase range between breakpoints at 1 rad/s and high frequency: -135°/decade
- Higher frequency: Phase shift is approaching -270°.

The sketches of the Bode plots are drawn in the figures below.



Note the results of the Bode plots simulation of the system in MATLAB are as shown in the figures below. The simulation results confirmed the sketches made.



b. As indicated in the frequency response plots, the gain margin (GM) and phase margin (PM) of the system are 3.52 dB at 0.707 rad/s and 11.4° at 0.572 rad/s.

c. The system is found to be stable as both the gain margin and the phase margin are positive. As these margins are relatively small, the system is very reactive (its transient response is oscillatory in nature) and could turn into instability easily in practice.

d. The transfer function of the forward path of the system is:

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(j\omega+1)(2j\omega+1)} = \frac{K}{-3\omega^2 + j\omega(1-2\omega^2)}$$

This open-loop transfer function has no poles in the right-half s plane. Thus, for stability, the $-1 + j0$ point should not be encircled by the Nyquist plot.

Let us find the point where the Nyquist plot crosses the negative real axis. Let the imaginary part of $G(j\omega)H(j\omega)$ be zero:

$$1 - 2\omega^2 = 0 \qquad \text{or} \qquad \omega = \pm\frac{1}{\sqrt{2}}$$

Substituting $\omega = 1/\sqrt{2}$ into $G(j\omega)H(j\omega)$, we obtain:

$$G\left(j\frac{1}{\sqrt{2}}\right)H\left(j\frac{1}{\sqrt{2}}\right) = \frac{K}{-3\left(\frac{1}{\sqrt{2}}\right)^2 + j\left(\frac{1}{\sqrt{2}}\right)\left[1-2\left(\frac{1}{\sqrt{2}}\right)^2\right]} = -\frac{2K}{3}$$

The critical value of the gain K is obtained by equating $-2K/3$ to -1:

$$-\frac{2}{3}K = -1 \qquad \text{or} \qquad K = \frac{3}{2}$$

The system is stable if $0 < K < 3/2$.

e. Based on the result in part (d), the system is stable if $0 < K < 3/2$. When the gain of the system $K = 2$, the system is unstable as it is not within the range of stable values of the gain of the system.

To improve the stability system, we could perform the following approaches:

- Reduce the gain of the system K to be below 3/2.
- Alternatively, introduce a derivative controller or a lead compensator in addition to the proportional controller already installed in the system. This type of controller or compensator introduce additional gain or phase margins of the system. As a result, the system will have sufficient gain and phase margins to be stable.

E. Root Locus Diagram

10. Consider a control system with the open loop transfer function as shown below.

$$G(s) = \frac{K}{(s+3)(s^2+2s+10)}$$

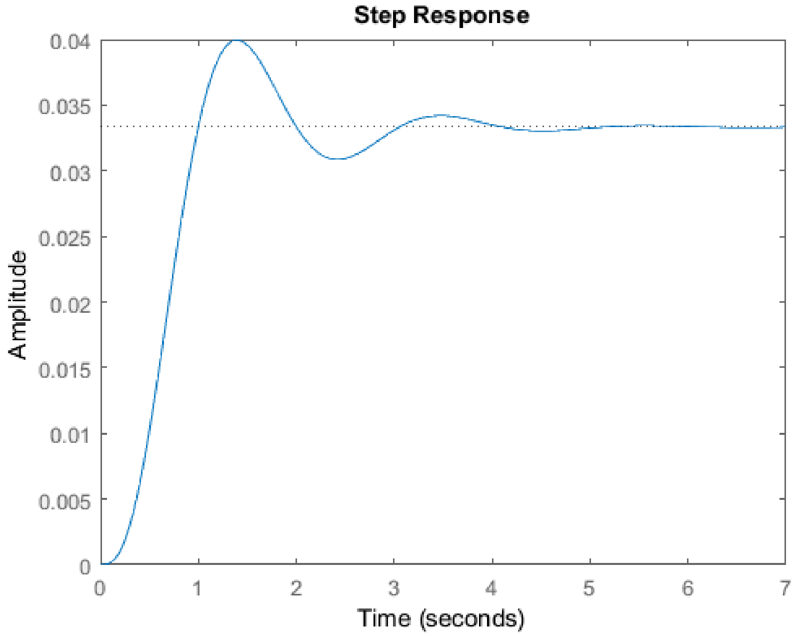
a. Sketch the root locus diagram of the system given above. [10 marks]

b. Determine the intercept point and angle of the asymptotes. [5 marks]

c. Determine the angles of departure from complex pole. [15 marks]

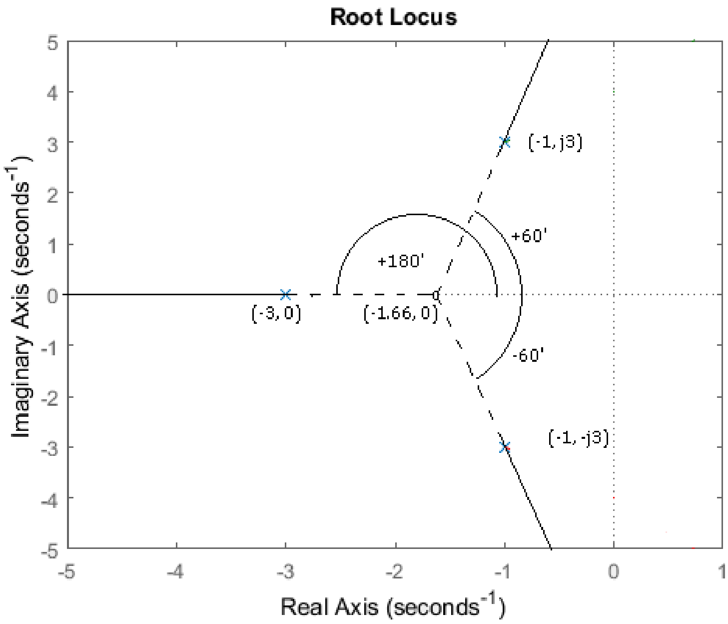
d. Calculate the value of the gain of the system (K) when the root locus crosses the imaginary axis (y-axis). [15 marks]

e. When the step response of the system is as shown in the figure given below, suggest a controller or compensator that could reduce the oscillatory behaviour of the system. [5 marks]

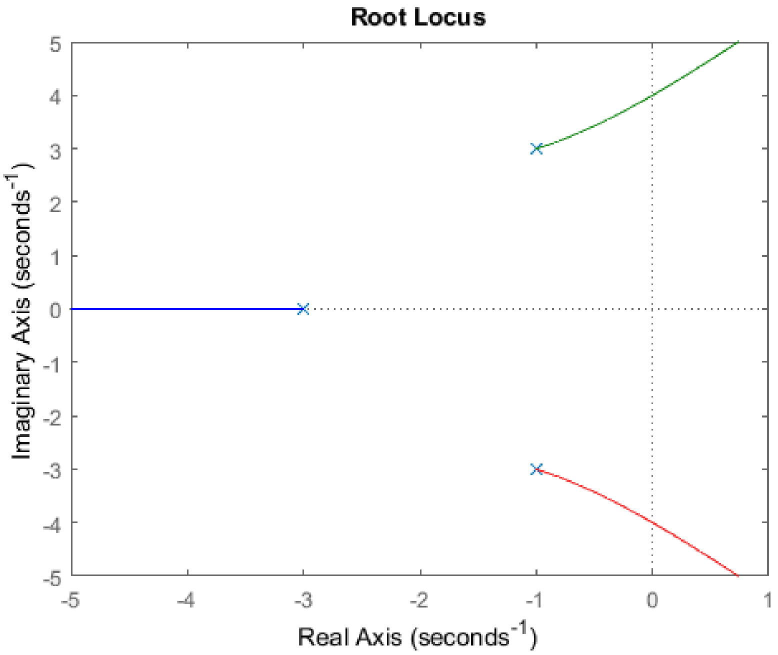


Solution

a. Sketch the root locus diagram of the system.



The result of root locus simulation of the system is shown in the figure below.



b. The intercept point and angle of the asymptotes are found from:

$$G(s) = \frac{1}{(s+3)(s^2+2s+10)}$$

Intercept point:

$$\sigma_{asymptote} = \frac{\sum_{n=1}^{k}(s+p_n) - \sum_{n=1}^{k}(s+z_n)}{\#n_p - \#n_z}$$

$$\frac{[(-3)+(-1)+(-1)]}{4-1} = -5/3$$

Angle of asymptotes:

$$\theta_{asymptote} = \pm\frac{(2k+1)\pi}{\#n_p - \#n_z} \qquad (k = 0, \pm 1, \pm 2, \dots)$$

When $k = 0$,

$$\theta_{asymptote} = \frac{\pi}{(3-0)} = \frac{\pi}{3}$$

When $k = 1$,

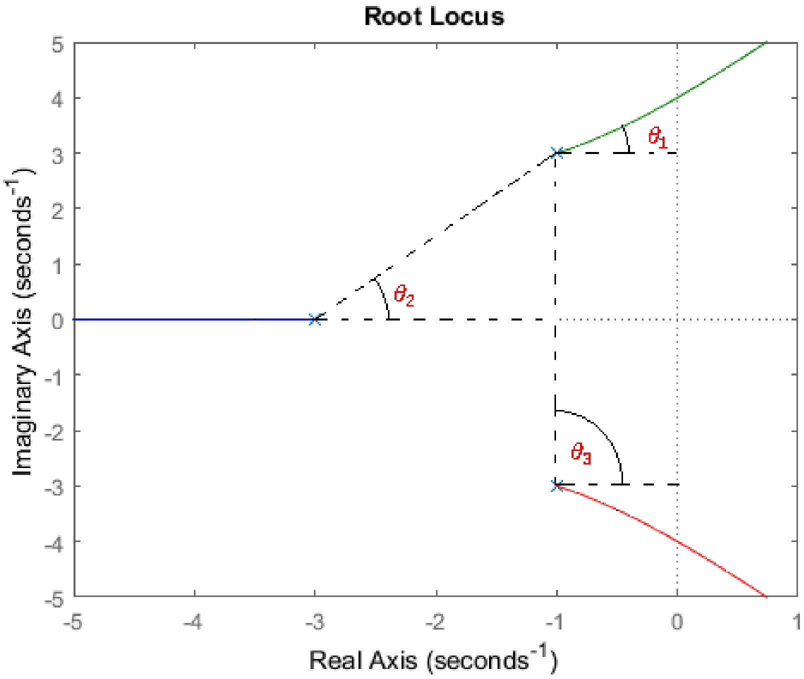
$$\theta_{asymptote} = \frac{(2+1)\pi}{(3-0)} = \pi$$

When $k = -1$,

$$\theta_{asymptote} = \frac{(-2+1)\pi}{(3-0)} = -\frac{\pi}{3}$$

c. Angle of departure from complex poles (θ_1) taken at $s = -1 + j3$ is calculated as follows:

$$G(s) = \frac{1}{(s+3)(s^2+2s+10)}$$



For a pole at $s = -3$

$$\theta_2 = \tan^{-1}\left(\frac{3}{2}\right) = 56.31°$$

For a pole at $s = -1 \pm j3$

$$\theta_3 = \tan^{-1}\left(\frac{6}{0}\right) = 90°$$

Using the following equation for all the angles of the poles and zeros and equating them to 180°.

$$-\theta_1 - \theta_2 - \theta_3 = 180°$$

Entering the values of the angles into the equation above:

$$-\theta_1 - 56.31° - 90° = 180°$$

As a result, the angle of departure from complex poles (θ_1) is:

$$\theta_1 = 33.69°$$

d. The value of system gain (K) when the root locus crosses the imaginary axis (y-axis) are:

$$G(s) = \frac{K}{(s+3)(s^2+2s+10)}$$

The closed loop transfer function of the unity feedback control system is determined from the following equation:

$$T(s) = \frac{G(s)}{1+G(s)}$$

Entering the transfer function of the control system into the equation above:

$$G(s) = \frac{\left[\frac{K}{(s+3)(s^2+2s+10)}\right]}{1+\left[\frac{K}{(s+3)(s^2+2s+10)}\right]} = \frac{K}{(s+3)(s^2+2s+10)+K}$$

Rearranging the equation given above:

$$G(s) = \frac{K}{s^3+5s^2+16s+(30+K)}$$

Analysis of the control system using Routh-Hurwitz criterion stability analysis:

s^3	1	16
s^2	5	$30+K$
s^1	$(50-K)/5$	
s^0	$30+K$	

From the stability analysis shown above, the value of the system gain (K) has to be between $0 < K < 50$.

e. The oscillatory behaviour of the system could be reduced by application of a PD controller or lead compensator. With careful tuning in of the given controller or compensator, you could reduce the percentage overshoot ($\%OS$) and hence also peak time or settling time in the transient response of the system.

11. Consider a control system with a transfer function of the plant ($G(s)$) as shown below:

$$G(s) = \frac{(s-2)(s-3)}{(s+1)(s+6)}$$

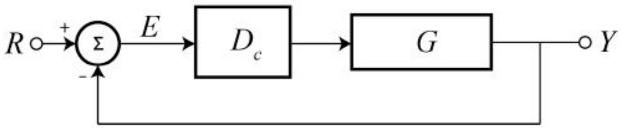
a. Sketch the root locus diagram of the given system. [10 marks]

b. Calculate the break-away and break-in points of the root locus.

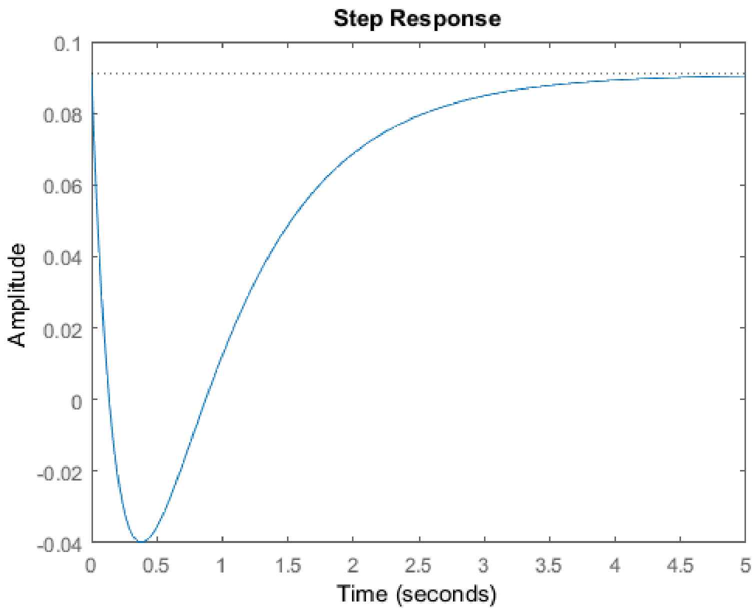
i. With differentiation. [10 marks]

ii. Without differentiation. [10 marks]

c. When a proportional controller ($D_c = K$) is added in series with the plant, determine the value of K so that the system is stable. [15 marks]

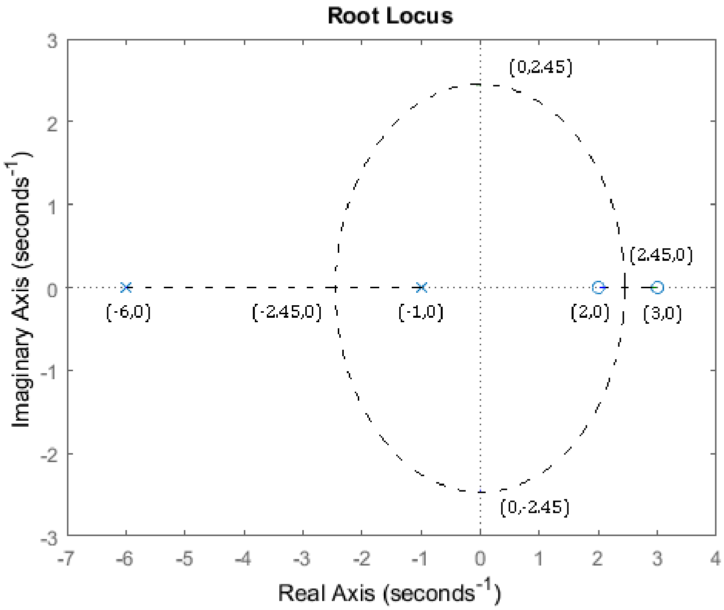


d. For $K = 0.1$, when the transient response of the system after it is subjected to a step input is as shown in the figure below, describe the two problems of the given control system. How could you improve the performance of the system? [5 marks]

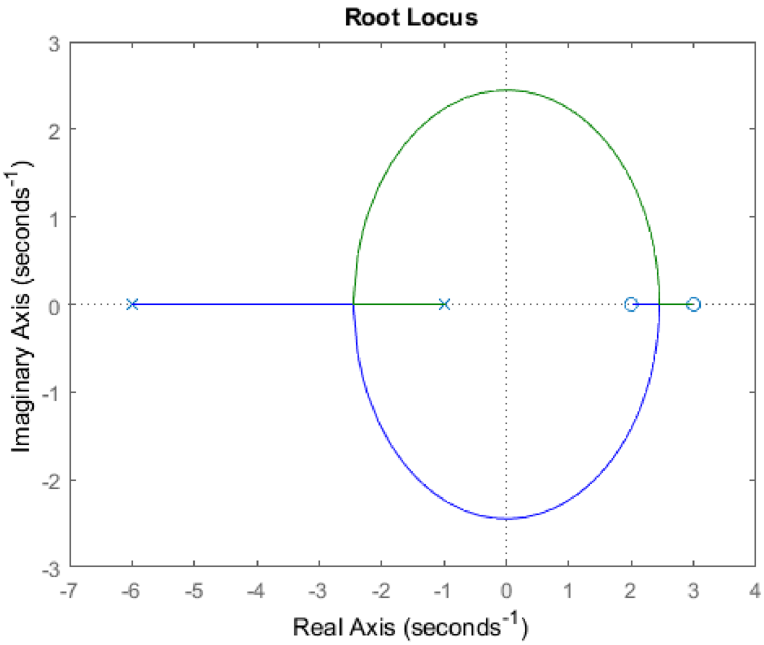


Solution

a. Root locus diagram of the given system is as shown in the figure below.



From the MATLAB simulation, the root locus of the system is as shown below.



b. The break-away and break-in points of the root locus:

$$G(s) = \frac{(s-2)(s-3)}{(s+1)(s+6)}$$

i. With differentiation:

Along the root locus, $KG = -1$

$$\frac{K(s-2)(s-3)}{(s+1)(s+6)} = -1$$

Substitute $s = \sigma$

$$\frac{K(\sigma-2)(\sigma-3)}{(\sigma+1)(\sigma+6)} = -1$$

We can solve for K and obtain

$$K = -\left(\frac{\sigma^2+7\sigma+6}{\sigma^2-5\sigma+6}\right)$$

Finding the extrema by taking the first derivative of the expression for K and setting it to zero.

$$\frac{dK}{d\sigma} = -\left[\frac{(2\sigma+7)(\sigma^2-5\sigma+6)-(\sigma^2+7\sigma+6)(2\sigma-5)}{(\sigma^2-5\sigma+6)^2}\right]$$

$$= -\left[\frac{(2\sigma^3-3\sigma^2-23\sigma+42)-(2\sigma^3+9\sigma^2-23\sigma+30)}{(\sigma^2-5\sigma+6)^2}\right]$$

$$= -\left[\frac{12\sigma^2-72}{(\sigma^2+5\sigma+6)^2}\right]$$

Thus, equating the numerator in the above equation to zero, the value of the root is:

$$\sigma = \pm\sqrt{6} = \pm 2.45$$

As a result, the break-away point is located at (0, -2.45) and the break-in point is at (0, 2.45).

ii. Without differentiation:

$$\sum_{i=1}^{Z}\frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P}\frac{1}{\sigma_b - p_j}$$

Where: p_j and z_i are the pole and zero values of CG from Z total zeros and P total poles.

$$\frac{1}{\sigma_b+2}+\frac{1}{\sigma_b+3} = \frac{1}{\sigma_b-1}+\frac{1}{\sigma_b-6}$$

Rearranging the equation given above and substitute $\sigma_b = \sigma$, we obtained:

$$\frac{2\sigma+5}{(\sigma+2)(\sigma+3)} = \frac{2\sigma-7}{(\sigma-1)(\sigma-6)}$$

$$(2\sigma-5)(\sigma^2-7\sigma+6) = (2\sigma-7)(\sigma^2+5\sigma+6)$$

$$-12\sigma^2+72 = 0$$

Thus, equating σ in the above equation to zero, the value of the root is:

$$\sigma = \pm\sqrt{6} = \pm 2.45$$

As a result, the break-away point is located at (0, -2.45) and the break-in point is at (0, 2.45).

c. The value of K so the system is stable is calculated as follows.

The open-loop transfer function of the system:

$$G(s) = \frac{K(s-2)(s-3)}{(s+1)(s+6)}$$

The closed-loop transfer function of the system.

$$\begin{aligned} T(s) &= \frac{G(s)}{1+G(s)} \\ &= \frac{\left[\frac{K(s-2)(s-3)}{(s+1)(s+6)}\right]}{1+\left[\frac{K(s^2-5s+6)}{(s^2+7s+6)}\right]} \\ &= \frac{K(s^2-5s+6)}{K(s^2-5s+6)+(s^2+7s+6)} \\ &= \frac{K(s^2-5s+6)}{(K+1)s^2+(7-5K)s+(6K+6)} \end{aligned}$$

The gain of the system (K) when the root locus across the imaginary axis is.

s^2	$(K+1)$	$6K+6$
s^1	$(7-5K)$	
s^0	$6K+6$	

From the Routh table, $K = -1$ and $K = \frac{7}{5} = 1.4$. So, when $K = 1.4$, the locus crosses the imaginary axis.

Based on the Routh table, focusing on the first column to be positive, for stable system $K > -1$ and $K < 1.4$. So, the value of K so the system is stable is $0 > K > 1.4$.

d. As shown in the given transient response figure, the system suffers from two problems e.g. a slow or sluggish system (its settling time is about 5 seconds) and a large steady-state error (the steady-state value is found to be less than 0.1 which is only 10% of the intended final value).

The transient response performance of the system can be improved with addition of Proportional Derivative controller or Lead compensator.

The steady-state problem can be eliminated with inclusion of PI controller and could be minimised with the addition of Lag compensator into the system.

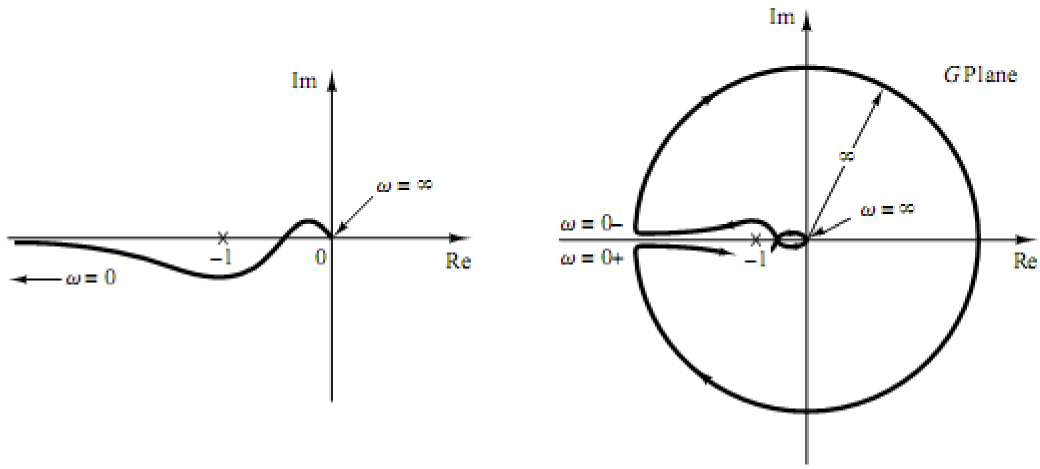
F. Nyquist Plot

12. The Nyquist plot (polar plot) of the open-loop frequency response of a unity-feedback control system is shown in the following diagram. Assuming that the Nyquist path in the s plane encloses the entire right-half s plane, draw a complete Nyquist plot in the G plane. Then answer the following questions:

    a. If the open-loop transfer function has no poles in the right-half s plane, is the closed-loop system stable? [5 marks]

    b. If the open-loop transfer function has one pole and no zeros in right-half s plane, is the closed-loop system stable? [5 marks]

    c. If the open-loop transfer function has one zero and no poles in the right-half s plane, is the closed-loop system stable? [5 marks]



Solution

Figure above shows a complete Nyquist plot in the G plane. The answers to the three questions are as follows:

a. The closed-loop system is stable, because the critical point $(-1 + j0)$ is not encircled by the Nyquist plot. That is, since $P = 0$ and $N = 0$, we have $Z = N + P = 0$.

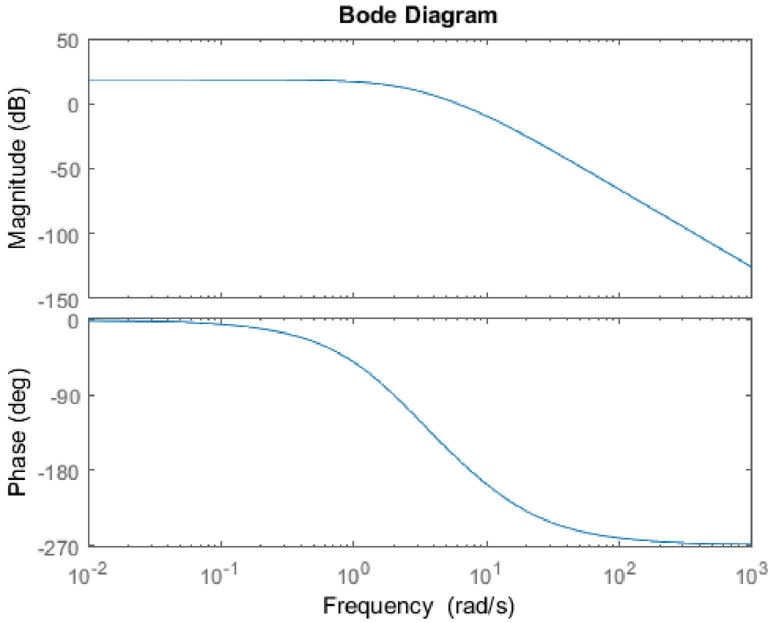
b. The open-loop transfer function has one pole in the right-half s plane, hence, $P = 1$. The open-loop system is unstable. For the closed-loop system to be stable, the Nyquist plot must encircle the critical point $(-1 + j0)$ once counterclockwise. However, the Nyquist plot does not encircle the critical point. Hence, $N = 0$. Therefore, $Z = N + P = 1$.The closed-loop system is unstable.

c. Since the open-loop transfer function has one zero, but no poles in the right-half s plane, we have $Z = N + P = 0$. Thus, the closed-loop system is stable. Note that the zeros of the open-loop transfer function do not affect the stability of the closed-loop system.

13. Given an open-loop control system as represented by the following transfer function.

$$G(s) = \frac{1}{(s+2)(s+3)(s+10)}$$

The Bode plots (magnitude and phase) of the system above are shown in the following figure given below:



a. From the Bode plots, sketch the Nyquist diagram of the system. [10 marks]

b. Determine the gain and phase margins from the sketch. [5 marks]

c. Based on the results of parts (a) and (b), describe the stability of the system. [5 marks]

d. With a help of a sample Nyquist diagram for illustration, describe how a proportional compensator could affect the Nyquist contour and stability of the system. [5 marks]

Solution

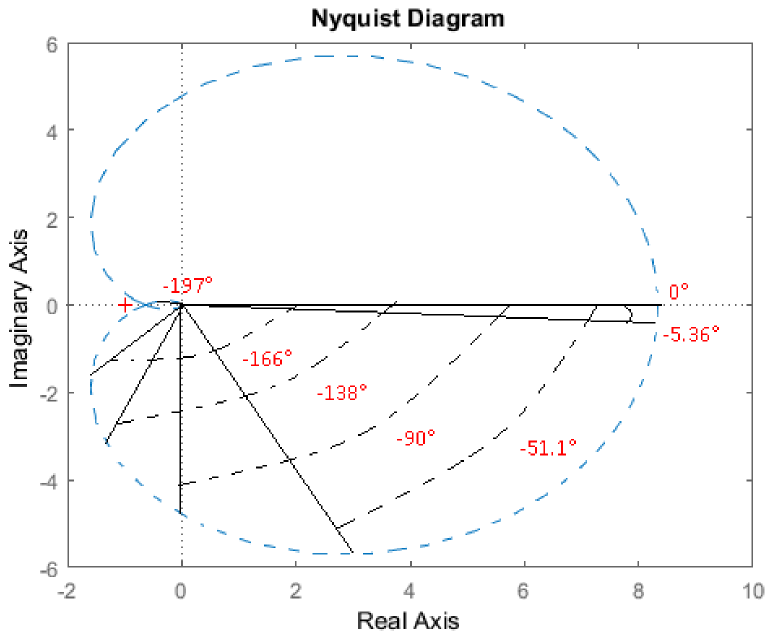
a. Sketch the Nyquist diagram of the system from the given Bode plots.

The frequency response points taken from the Bode plots are as shown in the table below.

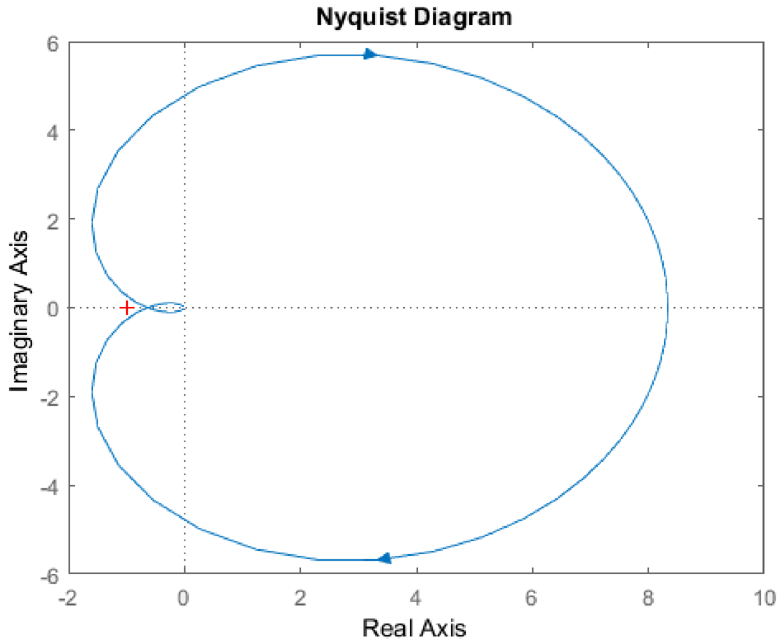
$\boldsymbol{\omega}$	$\lvert \boldsymbol{G(s)} \rvert$	$\angle \boldsymbol{G(s)}$
0.01	18.4 dB (8.317)	0°
0.1	18.4 dB (8.317)	-5.36°
1	17 dB (7.079)	-51.1°

2	13.6 dB (4.786)	-90°
4	6.35 dB (2.077)	-138°
6	0.1 dB (1.259)	-166°
10	-9.59 dB (0.332)	-197°

The sketch of the Nyquist diagram is as shown in the figure given below.



From MATLAB simulation, the Nyquist diagram is as shown in the figure below. The results from the simulation confirm the sketch.



b. The gain and phase margins are determined from the sketch.

$$Z = P + N$$

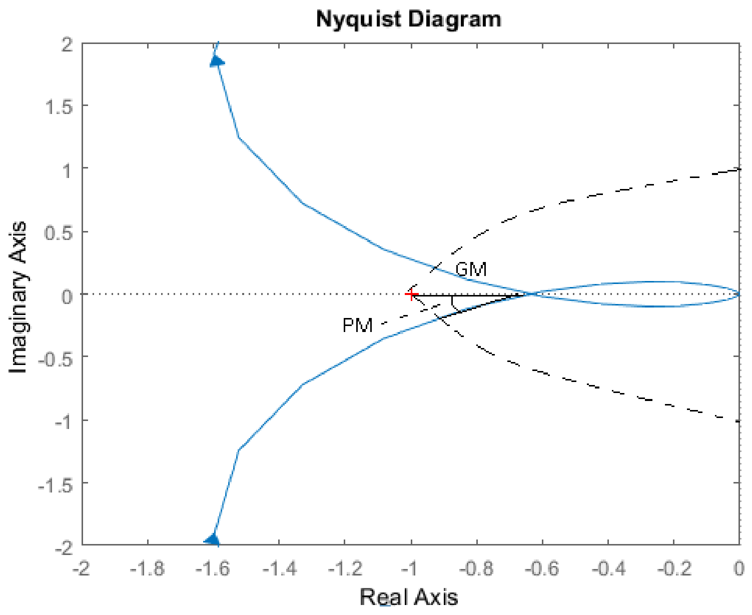
Note: Z = unstable closed-loop pole; P = unstable open-loop poles; N = # encirclement at the unity gain point (-1+j0).

$$\text{PM} = 180 + \arg[G(j\omega)(H(j\omega)]$$

and

$$\text{GM(in dB)} = 20\log[1/G(j\omega)H(j\omega)]$$

The gain and phase margins from the Nyquist is determined as shown in the figure below.

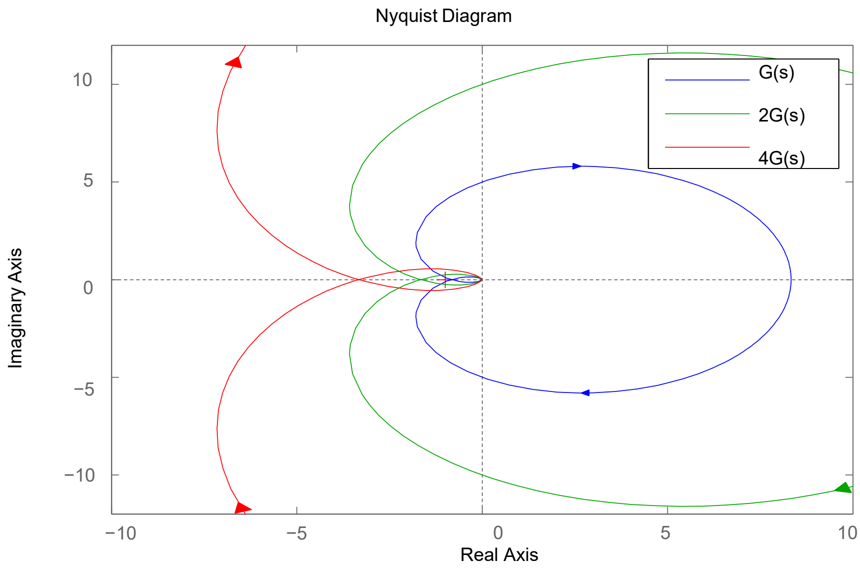


From the Nyquist diagram, the gain margin is +3.86 dB and the phase margins is +13.7°.

c. Based on the Nyquist diagram and values of gain and phase margins, the system is found to be stable.

- From Nyquist diagram – There is no encirclement of the contour at the test point (-1,0).
- From gain margin (GM) and phase margin (PM) determination – GM is found to be +3.86 dB and PM is found to be +13.7°. With this positive gain and phase margins, the system is stable.

d. A proportional compensator could affect contour in the Nyquist diagram by scaling it according to the value of the gain of the compensator. As a result, it could change the stability of the system because of this matter.



Too much proportional gain introduced to the system make the closed loop system to be unstable. As illustrated above the systems with the big values of gain have encirclements at the test point (-1,0) in the Nyquist diagram, hence the system becomes unstable.