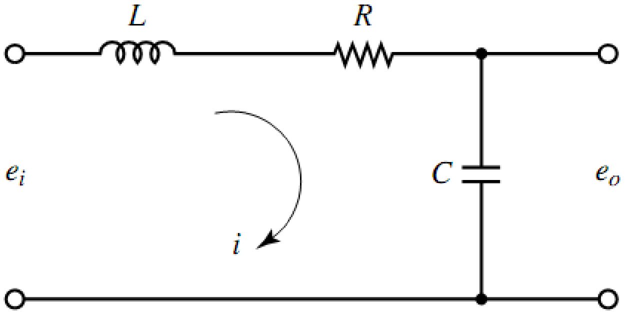
XMUT315 Control Systems Engineering

Mid-Term Test Revision Questions (Solution)

A. System Modelling

1. Consider the electrical circuit shown in figure below. The circuit consists of an inductance L (Henry), a resistance R (Ohm), and a capacitance C (Farad).



   a. Describe three types of modelling techniques in control systems. [3 marks]

   b. Determine the system's transfer function. [10 marks]

   c. If L = 1 mH, R = 1 kΩ, and C = 10 µF, calculate the roots of the characteristics equation of the system. Predict the time response of the system. [6 marks]

Solution

a. Types of modelling approaches:

- A scale physical model: proportional to the actual model.
- Mathematical model: described as function and variable in mathematical equation.
- Numerical model: represented as a set of numbers to describe system characteristic and behaviour.

b. Applying Kirchhoff's voltage law to the system, we obtain the following equations that give a mathematical model of the circuit.

The input voltage is the voltage source which is also the sum of voltage drops across the inductor, resistor, and capacitor.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

The output voltage is across the voltage across the capacitor.

$$\frac{1}{C}\int i\,dt = e_o$$

A transfer-function model of the circuit can also be obtained by taking the Laplace transforms of equations given above, assuming zero initial conditions, we obtain:

The equation for the input voltage:

$$LsI(s) + RI(s) + \left(\frac{1}{C}\right)\frac{1}{s}I(s) = E_i(s)$$

The equation for the output voltage:

$$\left(\frac{1}{C}\right)\frac{1}{s}I(s) = E_o(s)$$

If E_i is assumed to be the input and E_o the output, then the transfer function of this system is found to be:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

c. When L = 1 mH, R = 1 kΩ, and C = 10 µF, the transfer function of the equation becomes:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(10^{-3})(10 \times 10^{-6})s^2 + (10^3)(10 \times 10^{-6})s + 1}$$

$$= \frac{1}{10^{-8}s^2 + 10^{-2}s + 1} = \frac{10^8}{s^2 + 10^6 + 10^8}$$

The characteristic equation of the system is:

$$s^2 + 10^6 + 10^8$$

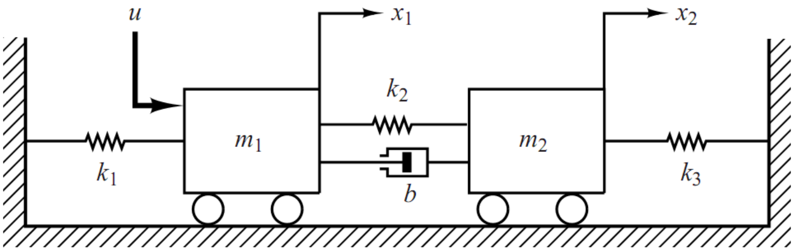
The roots of the characteristic equation of the system are:

$$s_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{10^6}{2(1)} \pm \frac{\sqrt{(10^6)^2 - 4(1)(10^8)}}{2(1)} = 5 \times 10^5 \pm 4.99 \times 10^5$$

Because we have two distinctive roots, thus the time response of the system is found to be overdamped (i.e. exponential).

2. You are given a mechanical system that consists of two interconnected moving carts as shown in the figure below. Note that u = force, m_1 = m_2 = masses, k_1 = k_2 = k_3 = spring constants, b = damper constant, and x_1 = x_2 = displacements. Assume zero initial conditions of the system.



a. Describe the significant of signal in control systems. [2 marks]

b. Obtain the transfer functions of the system above. [10 marks]

Solution

a. Significant of signals:

- Components are connected together by signals.
- Signals have many different forms.
- Must also have direction and name.
- Signals continue until interrupted.
- Signals and components are considered ideal.
- We add other signals and components to alter.

b. The equations of motion for the system shown in the figure above are evaluated as follows:

The forces acting in the first cart:

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 - k_2(x_1 - x_2) - b\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + u$$

Then, the forces acting in the second cart:

$$m_2\frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2 - x_1) - b\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

Simplifying, we obtain:

$$m_1\frac{d^2x_1}{dt^2} + b\frac{dx_1}{dt} + (k_1 + k_2)x_1 = b\left(\frac{dx_2}{dt}\right) + k_2x_2 + u$$

Then,

$$m_2\frac{d^2x_2}{dt^2} + b\frac{dx_2}{dt} + (k_2 + k_3)x_2 = b\left(\frac{dx_1}{dt}\right) + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain:

$$m_1s^2 + bs + (k_1 + k_2)X_1(s) = (bs + k_2)X_2(s) + U(s)$$

Then,

$$m_2s^2 + bs + (k_2 + k_3)X_2(s) = (bs + k_2)X_1(s)$$

Solving the first equation above for $X_2(s)$ and substituting it into the second equation above and simplifying, we get:

$$[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]X_1(s)$$
$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

From which we obtain:

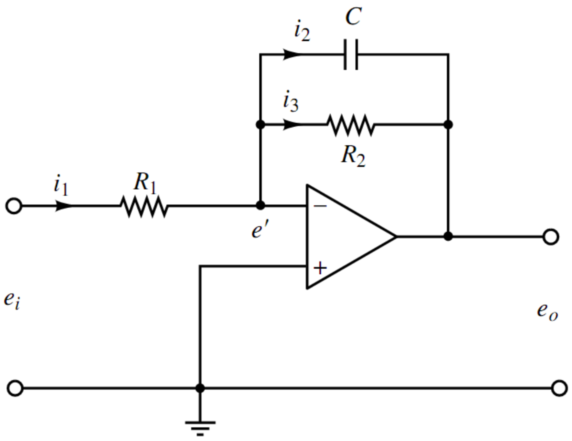
$$\frac{X_1(s)}{U(s)} = \frac{(m_2s^2 + bs + k_2 + k_3)}{[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]}$$

From the equations given above, we have:

$$\frac{X_2(s)}{U(s)} = \frac{(bs + k_2)}{[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2]}$$

Equations $X_1(S)/U(s)$ and $X_2(s)/U(s)$ are the transfer functions of the system respectively.

3. Figure below shows an electrical circuit involving an operational amplifier.



a. What are the two main goals of modelling physical systems? [2 marks]

b. Obtain the transfer function equation of the circuit $e_o(t)/e_i(t)$. [10 marks]

Solution

a. Goals of modelling physical systems:

- Develop mathematical models, i.e. ordinary differential equations that describe the relationship between input and output characteristics of a system.
- These equations can then be used to forecast the behaviour of the system under specific conditions.

b. Noting that the current flowing into the amplifier is negligible, we have:

$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = C\frac{d(e' - e_o)}{dt} \qquad i_3 = \frac{e' - e_o}{R_2}$$

Hence

$$i_1 = i_2 + i_3$$

Since we have:

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e' - e_o}{R_2}$$

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have:

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2}$$

Which can be written as:

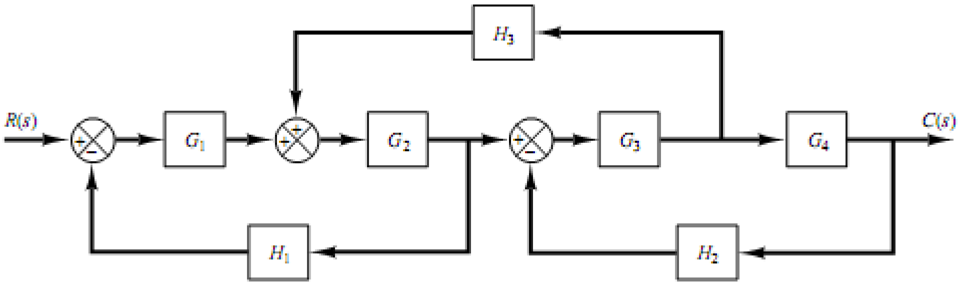
$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s)$$

The op-amp circuit shown in the figure above is a first-order lag circuit.

$$\frac{E_o(s)}{E_i(s)} = -\left(\frac{R_2}{R_1}\right)\frac{1}{R_2Cs + 1}$$

B. Feedback Control Systems

4. Given a physical system represented as a block diagram given below.

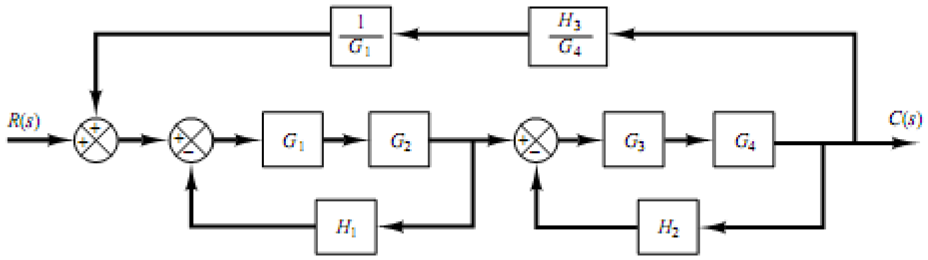


   a. Simplify the block diagram shown in figure below. Then obtain the closed-loop transfer function $C(s)/R(s)$. [10 marks]

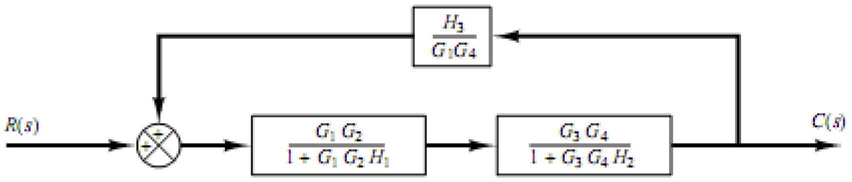
   b. The blocks H_1, H_2, and H_3 are identified as compensators. Why do we need compensators in control systems? [2 marks]

Solution

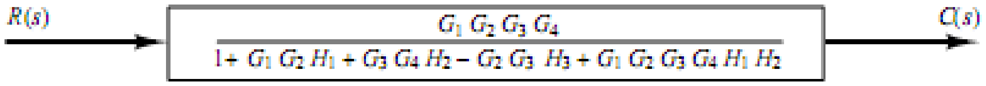
a. First move the branch point between G_3 and G_4 to the right-hand side of the loop containing G_3, G_4, and H_2. Then move the summing point between G_1 and G_2 to the left-hand side of the first summing point. See figure below.



By simplifying each loop, the block diagram can be modified as shown in the figure below.



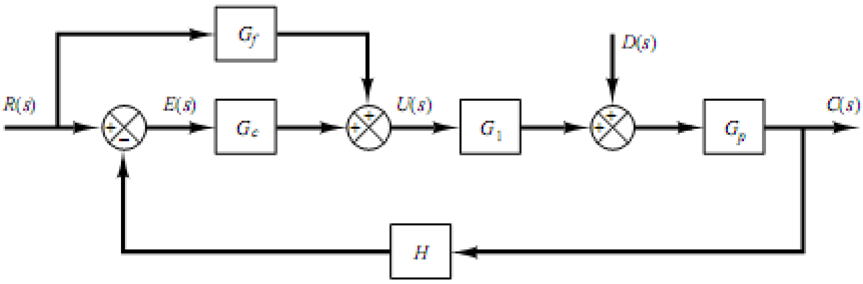
Further simplification results in figure below, from which the closed-loop transfer function $C(s)/R(s)$ is obtained as:



b. Compensator is needed as:

- In order to obtain the desired performance of the system, we use compensators. Compensators are applied to the system in the form of feed forward path gain adjustment.
- Compensate an unstable system to make it stable.
- A compensating network is used to minimize overshot.
- Compensators could increase the steady state accuracy of the system. An important point to be noted here is that the increase in the steady state accuracy brings instability to the system.
- Compensator could also introduce poles and zeros in the system thereby causes changes in the transfer function of the system. Due to this, performance specifications of the system change.

5. Referring to the system shown in the figure below, perform the following tasks.



a. Obtain transfer functions $C(s)/R(s)$ and $C(s)/D(s)$. [10 marks]

b. What are the things required for analysing a system? [2 marks]

Solution

a. From figure given above we have:

$$U(s) = G_f R(s) + G_e E(s)$$

$$C(s) = G_p[D(s) + G_1 U(s)]$$

$$E(s) = R(s) - HC(s)$$

By substituting $U(s)$ into $C(s)$, we get:

$$C(s) = G_p D(s) + G_1 G_p\left[G_f R(s) + G_e E(s)\right]$$

By substituting $C(s)$ into Equation $E(s)$, we obtain:

$$C(s) = G_p D(s) + G_1 G_p\left\{G_f R(s) + G_e[R(s) - HC(s)]\right\}$$

Solving this last equation for $C(s)$, we get:

$$C(s) + G_1 G_p G_e HC(s) = G_p D(s) + G_1 G_p\left(G_f + G_e\right)R(s)$$

Hence

$$C(s) = \frac{G_p D(s) + G_1 G_p\left(G_f + G_e\right)R(s)}{1 + G_1 G_p G_e H}$$

Note that equation above gives the response $C(s)$ when both reference input $R(s)$ and disturbance input $D(s)$ are present. To find transfer function $C(s)/R(s)$, we let $D(s) = 0$ in the equation. Then we obtain:

$$\frac{C(s)}{R(s)} = \frac{G_1 G_p\left(G_f + G_e\right)}{1 + G_1 G_p G_e H}$$

Similarly, to obtain transfer function $C(s)/D(s)$, we let $R(s) = 0$ in the equation. Then $C(s)/D(s)$ can be given by:

$$\frac{C(s)}{D(s)} = \frac{G_p}{1 + G_1 G_p G_e H}$$

c. In order to analyse a system:

- We identify an input signal [a variable].
- Using block diagram components [basic block, summing junction, and take-off point].
- We combine internal signals [modified variables].
- To produce the output signal [another variable].

Then, the input-output relationship may then be determined.

C. Stability Analysis

6. Apply Routh Hurwitz stability criterion for system given in the following equation:

$$s^3 + 2s^2 + s + 2 = 0$$

a. Determine stability and the roots of the system. [10 marks]

b. Compare the stability and roots of the system in part (a) with the following system: [5 marks]

$$s^3 - 3s + 2 = 0$$

Solution

a. Applying Routh Hurwitz stability criterion, the array of coefficients of the system given above is as follows:

s^3	1	1	+
s^2	2	2	+
s^1	$0 \approx \epsilon$		+
s^0	2		+

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Thus, the equation given above has two roots at $s = \pm j$.

b. Comparing with the system in part (a) with the system in part (b), if the sign of the coefficient above the zero (ϵ) is opposite that below it as in the system in part (b), it indicates that there is one sign change. For example, for the system in part (b) with the equation:

$$s^3 - 3s + 2 = (s-1)^2(s+2) = 0$$

The array of coefficients is:

s^3	1	-3	+
s^2	$0 \approx \epsilon$	2	+

s^1	$-3-\frac{2}{\epsilon}$		-
s^0	2		+

There are two sign changes of the coefficients in the first column. So, there are two roots in the right-half s plane.

This agrees with the correct result indicated by the factored form of the polynomial equation.

7. Given a control system that is represented by the following equation:

$$s^5+2s^4+24s^3+48s^2-25s-50=0$$

a. By applying Routh Hurwitz criterion, evaluate the stability of a system. [10 marks]

b. Determine the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis. [5 marks]

Solution

a. Using Routh Hurwitz criterion, the array of coefficients of the system given above is:

s^5	1	24	-25	
s^4	2	48	-50	Auxiliary polynomial ($P(s)$)
s^3	0	0		
$\cdots$				

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd-numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial $P(s)$ is:

$$P(s)=2s^4+48s^2-50$$

Which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis).

These pairs are obtained by solving the auxiliary polynomial equation $P(s)=0$. The derivative of $P(s)$ with respect to s is:

$$\frac{dP(s)}{ds}=8s^3+96s$$

The terms in the s^3 row are replaced by the coefficients of the last equation—that is, 8 and 96. The array of coefficients then becomes:

s^5	1	24	-25		+
s^4	2	48	-50		+
s^3	8	96		Coefficients of $dP(s)/ds$	+
s^2	24	-50			+
s^1	112.7	0			+
s^0	-50				-

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part (e.g. pole at the right-hand side of the s-plane that could lead to an unstable system).

b. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

We obtain:

$$s^2 = 1 \qquad \text{and} \qquad s^2 = -25$$

Or

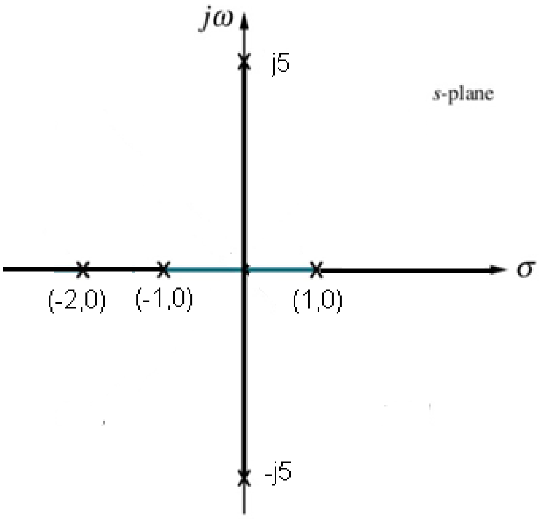
$$s = \pm 1 \qquad \text{and} \qquad s = \pm j5$$

These two pairs of roots of $P(s)$ are a part of the roots of the original equation.

As a matter of fact, the original equation can be written in factored form as follows:

$$(s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0$$

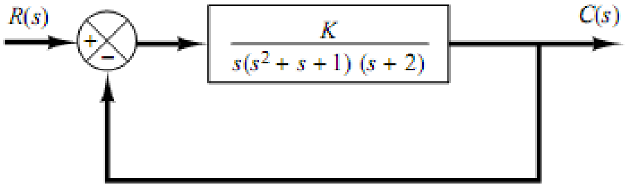
Clearly, the original equation has one unstable root with a positive real part (e.g. 1).



The rest of the roots are the stable pair of the unstable root at -1, a pair of complex roots at the y-axis (e.g. j5 and -j5) and a root at -2.

Finally, a row of zeros in the Routh-Hurwitz table indicates two pairs of real and imaginary roots symmetrical about the x-and y-axes respectively.

8. Consider the system shown below.



The closed-loop transfer function of the system is given as:

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

a. Determine the range of K for stability. [10 marks]

b. Define stability of the system by contrasting stable condition with unstable condition. [2 marks]

Solution

a. For given system, the characteristic equation of the system is:

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Applying Routh-Hurwitz criteria, the array of coefficients becomes:

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	
s^1	$2 - 9/7K$		
s^0	K		

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

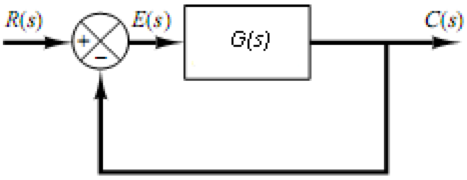
$$14/9 > K > 0$$

When $K = 14/9$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude. Note that the ranges of design parameters that lead to stability may be determined by use of Routh-Hurwitz's stability criterion.

b. A linear, time-invariant system is *stable* if: the natural response approaches zero as time approaches infinity. A linear, time-invariant system is *unstable* if: the natural response grows without bound as time approaches infinity.

D. Time Responses & Steady State Analysis

9. Consider the second order system as shown in the figure given below.



a. If the transfer function of the plant $G(s)$ is as given below, determine the damping ratio (ζ) and natural frequency of the system (ω_n). [5 marks]

$$G(s) = \frac{100}{4s^2 + 24s + 100}$$

b. Obtain the rise time (T_r), peak time (T_p), maximum overshoot (M_p), and settling time (T_s) when the system is subjected to a unit-step input. [10 marks]

Solution

a. Knowing the transfer function of the standardized second order system is given as:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

So, rearranging the equation of the given system into the standardized second order form, the transfer function of the system becomes:

$$G(s) = \frac{25}{s^2 + 6s + 25}$$

The natural frequency (ω_n) of the system is:

$$\omega_n^2 = 25 \qquad \text{so} \qquad \omega_n = 5$$

The damping ratio (ζ) of the system is:

$$2\zeta\omega_n = 6 \qquad \text{so} \qquad \zeta = \frac{6}{(2)(5)} = 0.6$$

Thus, the damping ratio and natural frequency of the system are ζ = 0.6 and ω_n = 5 rad/sec respectively.

b. From the values of ζ and ω_n obtained in part (a), we could calculate the other parameters of transient response of the system:

Rise Time (T_r):

The rise time is:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{3.14 - \phi}{5\sqrt{1-(0.6)^2}}$$

where ϕ is given by:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{0.8}{0.6}\right) = 0.93 \text{ rad}$$

The rise time T_r is thus:

$$T_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak Time (T_p):

The peak time is measured as the first peak (n = 1):

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{(1)(3.14)}{5\sqrt{1-(0.6)^2}} = 0.785 \text{ sec}$$

Maximum Overshoot (M_p):

The maximum overshoot is the percentage overshoot of the first peak:

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} * 100\% = e^{-\left(\frac{(0.6)(3.14)}{\sqrt{1-(0.6)^2}}\right)} * 100\% = 0.095 * 100\% = 9.5\%$$

The maximum percent overshoot is thus 9.5%.

Settling Time (T_s):

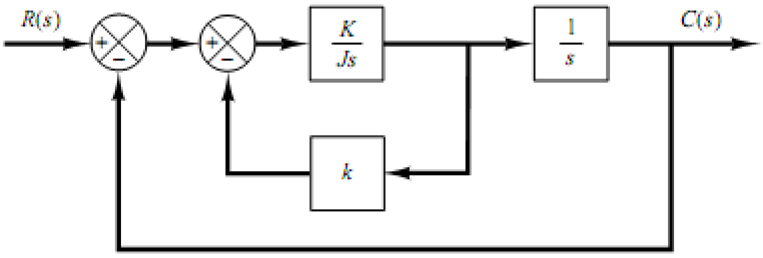
For the 2% criterion, the settling time is:

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(5)(0.6)} = 1.33 \text{ sec}$$

For the 5% criterion, the settling time is:

$$T_s = \frac{3}{\zeta\omega_n} = \frac{3}{(5)(0.6)} = 1 \text{ sec}$$

10. Given a system in the figure below.



a. Determine the values of K and k of the closed-loop system shown in the figure above, so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J = 1 kg-m^2. [10 marks]

b. What is the time response of a system when its poles are moved along a constant radial line? [2 marks]

Solution

a. The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Kks + K}$$

By substituting J = 1 kg-m^2 into this last equation, we have:

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + Kks + K}$$

Note that in this problem:

$$\omega_n = \sqrt{K} \qquad \text{and} \qquad 2\zeta\omega_n = Kk$$

The maximum overshoot (M_p) is the percentage overshoot of the first peak (n = 1):

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Which is specified as 25%. Hence

$$e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.25$$

From which after taking natural log on both sides:

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.386$$

Or

$$\zeta = 0.404$$

The peak time (T_p) is specified as 2 sec and assuming it is the first peak (n = 1):

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = 2$$

Then, the undamped natural frequency (ω_n) is:

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{3.14}{2\sqrt{1-(0.404)^2}} = 1.72$$

Therefore, we obtain the values of K and k respectively:

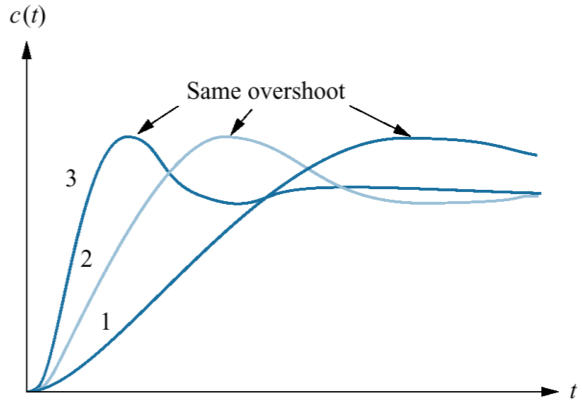
$$K = \omega_n^2 = (1.72)^2 = 2.95 \text{ Nm}$$

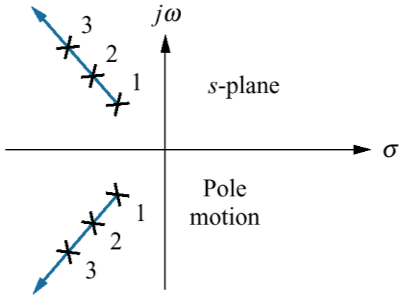
And

$$k = \frac{2\zeta\omega_n}{K} = \frac{2(0.404)(1.72)}{2.95} = 0.471 \text{ sec}$$

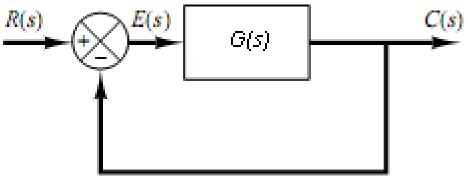
b. Moving the poles of the system along a constant radial line:

- Percent overshoot remains the same.
- The responses look exactly alike, except for their speed.
- The farther the poles are from the origin, the more rapid the response.





11. Consider the closed loop feedback system shown in the figure given below.



a. Derive the equation for steady-state error of the system. [4 marks]

b. Define steady-state error and describe three types of input used for testing steady-state condition of a system. [4 marks]

c. If the transfer function of the plant $G(s)$ is as stated below, calculate the static-error constants of the system (K_p, K_v, and K_a). [6 marks]

$$G(s) = \frac{s+5}{s(s+2)(s+10)}$$

d. Based on the results in part (c), calculate the steady-state error of the system whenever they are subjected to the following test inputs:

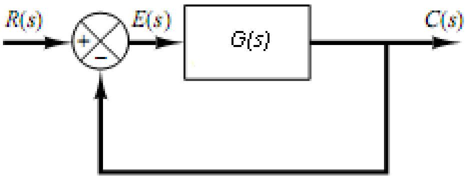
i. Step input. [2 marks]

ii. Ramp input. [2 marks]

iii. Parabolic input. [2 marks]

Solution

a. The derivation of the equation is as shown below:



Referring to the block diagram given above, the error of the system is found from the following equation:

$$E(s) = R(s) - C(s) \qquad (1)$$

The output of the system is found from the following equation:

$$C(s) = E(s)G(s) \qquad (2)$$

Combining both equations given above by substituting the equation (2) into the equation (1), the equation for the error of the system is given as:

$$E(s) = R(s) - E(s)G(s)$$

Thus, rearranging the above equation:

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Applying the final value theorem to the error equation to find the error of the unity feedback system at steady-state condition:

$$E(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\left[\frac{R(s)}{1 + G(s)}\right]$$

b. For a given system, the steady-state error is then defined as the difference between demanded and actual output when t -> ∞.

The steady-state error is now defined in the course for specific test inputs:

- Step input – a constant amplitude signal.
- Ramp input – a linear unary real function signal, whose graph is shaped like a ramp.
- Parabolic input – a quadratic function signal.

c. The static-error constants of the system are calculated as follows:

$$G(s) = \frac{s + 5}{s(s + 2)(s + 10)}$$

Proportional-error constant (K_p):

$$K_p = \lim_{s \to 0} G(s) = \frac{5}{(0)(2)(10)} = \infty$$

Velocity-error constant (K_v):

$$K_v = \lim_{s \to 0} sG(s) = \frac{5}{(2)(10)} = 0.25 \quad \text{(constant)}$$

Acceleration-error constant (K_a):

$$K_a = \lim_{s \to 0} s^2 G(s) = (0)\frac{5}{(2)(10)} = 0$$

d. Since the system is a Type 1 system as it has an integration in its characteristic equation, as a result, the steady-state errors of the system are calculated as follows.

Steady-state error for the step input ($e_{step}(\infty)$):

$$e_{step}(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 + \infty} = 0$$

Steady-state error for the ramp input ($e_{ramp}(\infty)$):

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{1}{0.25} = 4$$

Steady-state error for the parabolic input ($e_{parabola}(\infty)$):

$$e_{parabola}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$