VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

XMUT315 Control Systems Engineering

Tutorial 6: Analysis with Bode Plots (Solution)

1. You are given the following control systems as follows.
   i. System (i)

$$G_1(s) = \frac{1}{s+2}$$

   ii. System (ii)

$$G_2(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

   iii. System (iii)

$$G_3(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

a. Find the analytical expression and plots for the gain and phase of the frequency response of the systems above. [20 marks]

b. Describe each system in terms of its frequency response, transient characteristics, stability and steady-state behaviours. [20 marks]

Solution

a. Analytical expression and plots for the gain and phase of the frequency response of the systems.

   i. System (i), with its transfer function

$$G(s) = \frac{1}{s+2}$$

First substitute $s = j\omega$ in the system function and obtain:

$$G(j\omega) = \frac{1}{j\omega+2} = \frac{2-j\omega}{\omega^2+4}$$

The magnitude of this complex number is the gain frequency response.

$$|G(j\omega)| = \frac{1}{\sqrt{(\omega^2 + 4)}}$$

The phase angle of is the phase frequency response.

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

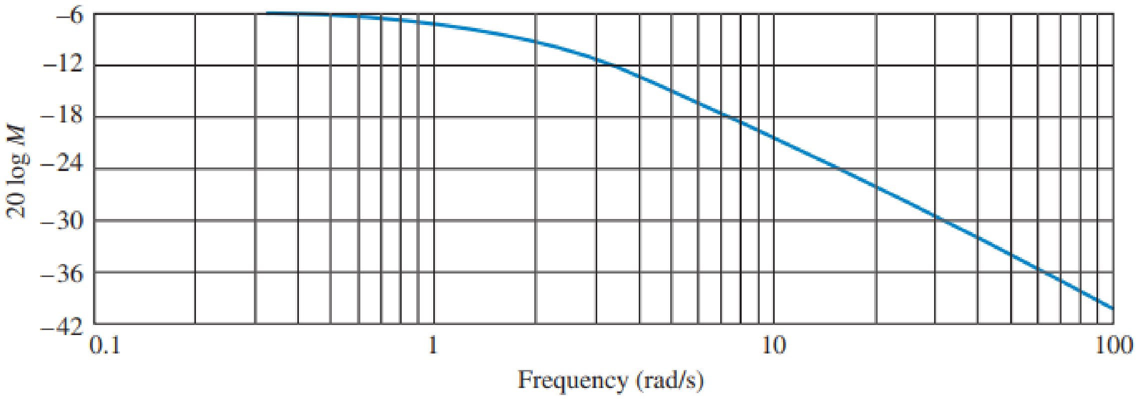
The system, $G(j\omega)$ can be plotted in two ways:

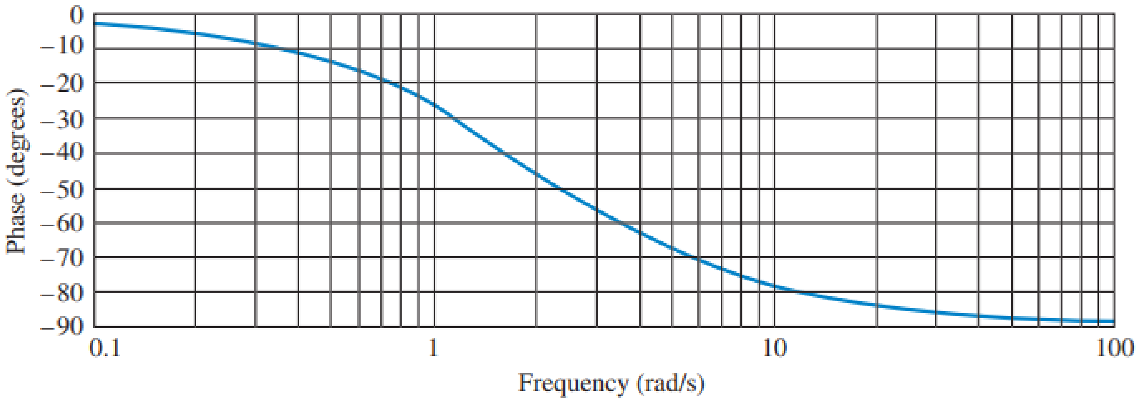
- Separate gain and phase plots.
- Polar plot.

The figure below shows separate gain and phase diagrams, where the gain diagram is $20\log|G(\omega)| = 20\log(1/\sqrt{\omega^2+4})$ vs. $\log\omega$, and the phase diagram is $\angle G(\omega) = -\tan^{-1}(\omega/2)$ vs. $\log\omega$.

ω	$\log\omega$	$20\log\lvert G(\omega)\rvert = 20\log(1/\sqrt{\omega^2+4})$	$\angle G(\omega) = -\tan^{-1}(\omega/2)$
0.1	-1	-6	-2.86°
1	0	-7	-26.57°
10	1	-20	-78.69°
100	2	-40	-88.85°

The gain and phase plots of the system are shown below.





The polar plot is a plot of magnitude and phase shift for different ω as shown in the following table.

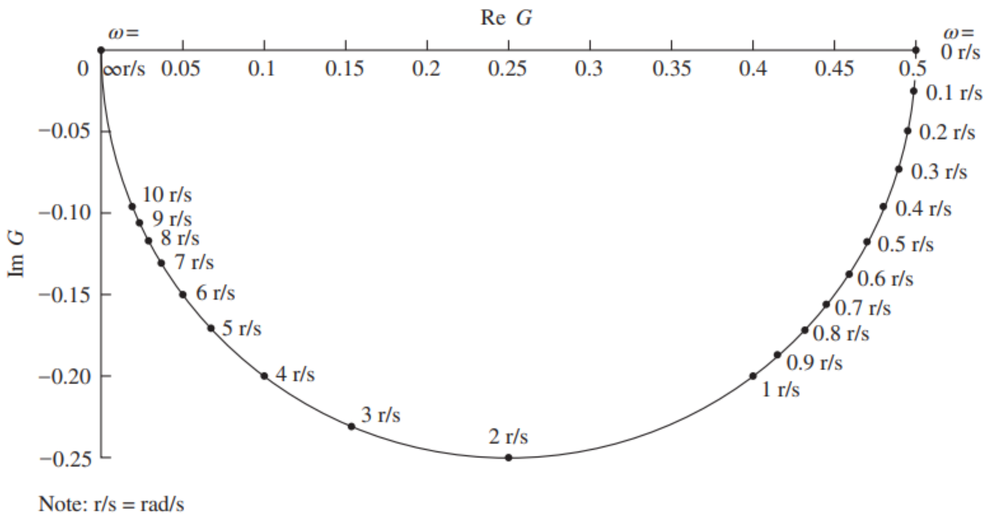
$$|G(\omega)| = \frac{1}{\sqrt{\omega^2 + 4}}$$

And

$$\angle G(\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

$\boldsymbol{\omega}$	$\mathbf{log}\,\boldsymbol{\omega}$	$\boldsymbol{\lvert G(\omega)\rvert = \frac{1}{\sqrt{\omega^2+4}}}$	$\boldsymbol{\angle G(\omega) = -\tan^{-1}(\omega/2)}$
0.1	-1	0.5	-2.86°
1	0	0.447	-26.57°
2	0.301	0.353	-45°
10	1	0.1	-78.69°
100	2	0.01	-88.85°

The polar plot of the system is given in the graph given below.



ii. System (ii), with its transfer function:

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

We will make a Bode plot for the open-loop function:

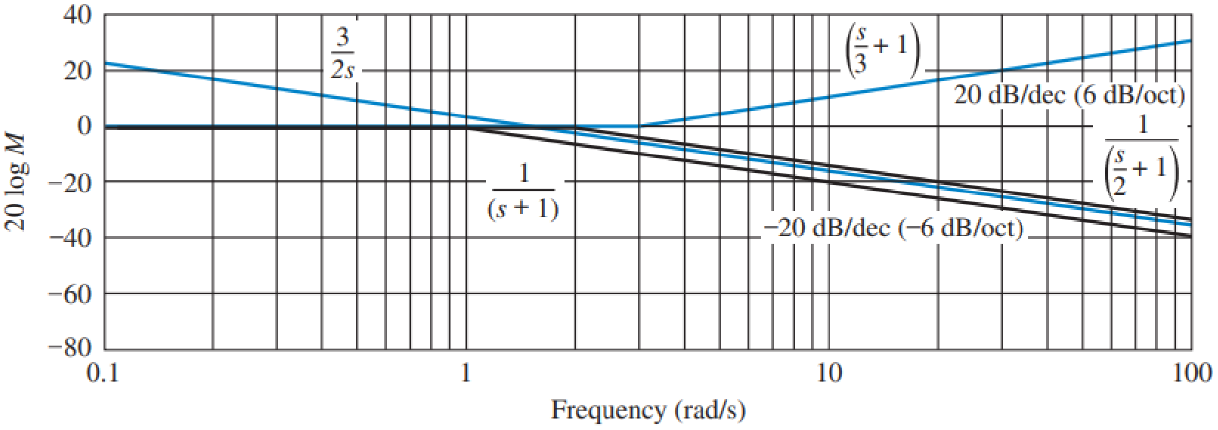
$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

The Bode plot is the sum of the Bode plots for each first-order term. Thus, it is convenient to use the normalized plot for each of these terms so that the low-frequency asymptote of each term, except the pole at the origin, is at 0 dB, making it easier to add the components of the Bode plot.

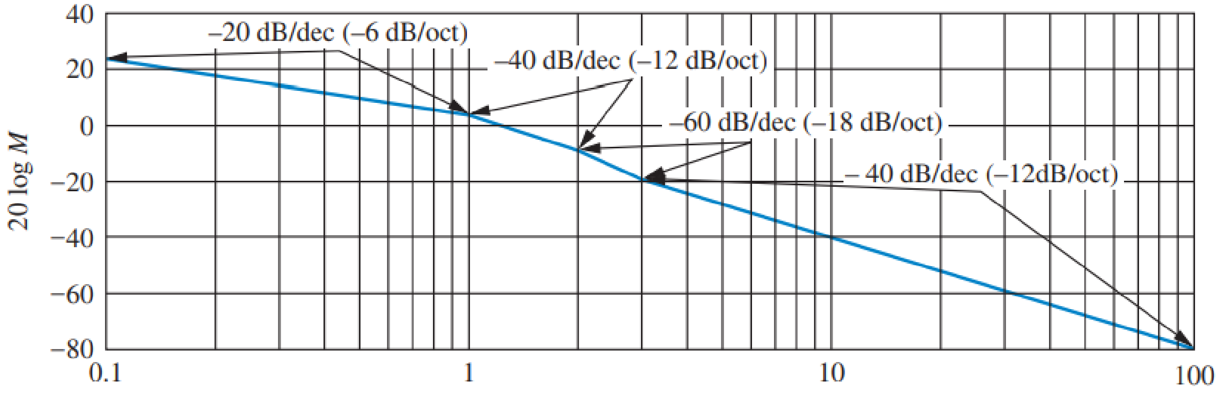
We rewrite $G(s)$ showing each term normalized to a low-frequency gain of unity. Hence,

$$G(s) = \frac{\frac{3}{2}K(\frac{s}{3}+1)}{s(s+1)\left(\frac{s}{2}+1\right)}$$

Now determine that the break frequencies are at 1, 2, and 3. The gain plot should begin a decade below the lowest break frequency and extend a decade above the highest break frequency.



(*a*)



(*b*)

Hence, we choose 0.1 radian to 100 radians, or three decades, as the extent of our plot.

At $\omega = 0.1$ the low-frequency value of the function is found from the previous equation using the low-frequency values for all the $(s/a) + 1$ terms (that is $s = 0$) and the actual value for the s term in the denominator. Thus,

$$G(j0.1) \approx \frac{\left(\frac{3}{2}\right)K}{0.1} = 15K$$

The effect of K is to move the gain curve up (increasing K) or down (decreasing K) by the amount of $20\log K$. K has no effect upon the phase curve. If we choose $K = 1$, the gain plot can be denormalized later for any value of K that is calculated or known.

Figure part (a) given above shows each component of the Bode log-gain of the frequency response. Summing the components yields the composite plot shown in part (b).

The results are summarized in the following table that shows Bode's gain plot i.e. slope contribution from each pole and zero, which can be used to obtain the slopes.

Description	**Frequency (rad/s)**			
	0.1 (start: pole at 0)	**1 (start: pole at -1)**	**2 (start: pole at -2)**	**3 (start: zero at -3)**
Pole at 0	-20	-20	-20	-20
Pole at -1	0	-20	-20	-20
Pole at -2	0	0	-20	-20
Zero at -3	0	0	0	20
Total slope (dB/dec)	-20	-40	-60	-40

Each pole and zero is itemized in the first column. Reading across the table shows its contribution at each frequency. The last row is the sum of the slopes and correlates with part (b) of the previous figure.

Gain Plot

The Bode gain plot for $K = 1$ starts at $\omega = 0.1$ with a value of $20 \log 15 = 23.52$ dB and decreases immediately at a rate of -20 dB/decade, due to the s term in the denominator.

At $\omega = 1$, the $s + 1$ term in the denominator begins its 20 dB/decade downward slope and causes an additional -20 dB/decade negative slope, or a total of -40 dB/decade.

At $\omega = 2$, the term $s/2 + 1$ begins its -20 dB/decade slope, adding yet another -20 dB/decade to the resultant plot, or a total of -60 dB/decade slope that continues until ω=3.

At this $\omega = 3$ frequency, the $s/3 + 1$ term in the numerator begins its positive 20 dB/decade slope. The resultant gain plot, therefore, changes from a slope of -60 dB/decade to -40 dB/decade at $\omega = 3$, and continues at that slope, since there are no other break frequencies.

The slopes are easily drawn by sketching straight-line segments decreasing by 20 dB over a decade.

For example, the initial -20 dB/decade slope is drawn from 23.52 dB at $\omega = 0.1$, to 3.52 dB (a 20 dB decrease) at $\omega = 1$.

The -40 dB/decade slope starting at $\omega = 1$ is drawn by sketching a line segment from 3.52 dB at $\omega = 1$, to -36.48 dB (a 40 dB decrease) at $\omega = 10$, and using only the portion from $\omega = 1$ to $\omega = 2$.

The next slope of -60 dB/decade is drawn by first sketching a line segment from $\omega = 2$ to $\omega = 20$ (1 decade) that drops down by 60 dB and using only that portion of the line from $\omega = 2$ to ω=3.

The final slope is drawn by sketching a line segment from $\omega = 3$ to $\omega = 30$ (1 decade) that drops by 40 dB. This slope continues to the end of the plot.

Phase Plot

Phase is handled similarly. However, the existence of breaks, a decade below and a decade above the break frequency, requires a little more bookkeeping.

The following table that is about Bode phase plot i.e. slope contribution from each zero and pole of the system. It shows the starting and stopping frequencies of the 45°/decade slope for each of the poles and zeros.

For example, reading across for the pole at -2, we see that the -45° slope starts at a frequency of 0.2 and ends at 20. Filling in the rows for each pole and then summing the columns yields the slope portrait of the resulting phase plot.

Looking at the row marked Total slope, we see that the phase plot will have a slope of -45°/decade from a frequency of 0.1 to 0.2.

The slope will then increase to -90°/decade from 0.2 to 0.3.

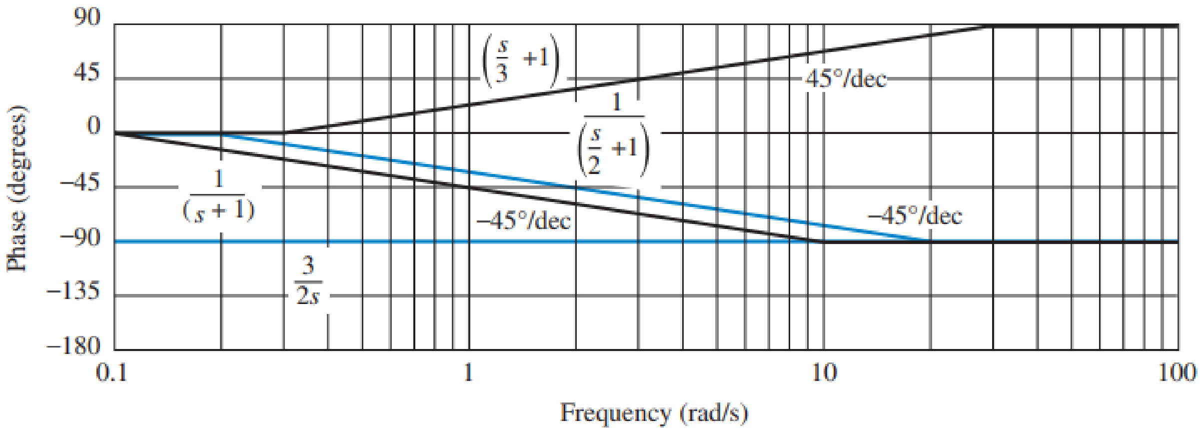
The slope will return to -45°/decade from 0.3 to 10 rad/s.

A slope of 0 ensues from 10 to 20 rad/s, followed by a slope of +45°/decade from 20 to 30 rad/s.

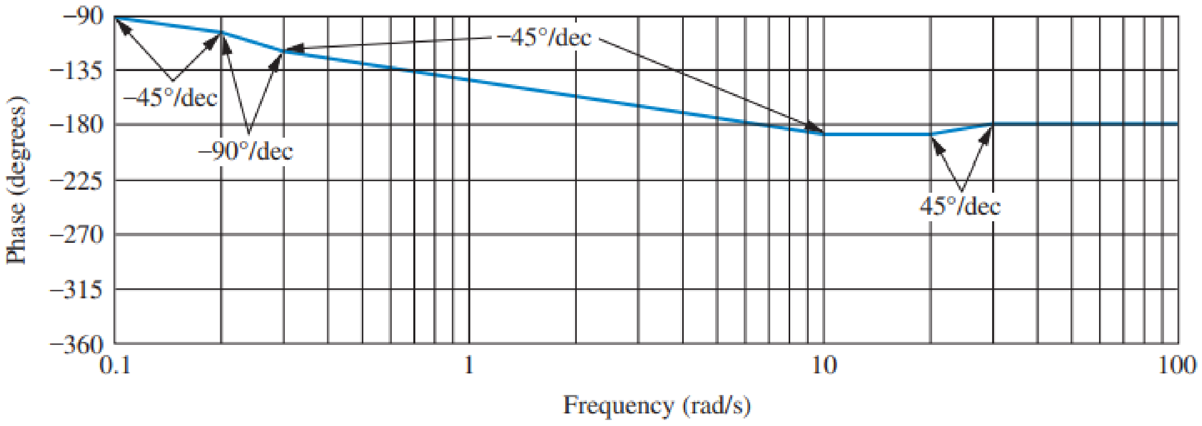
Finally, from 30 rad/s to infinity, the slope is 0°/decade.

Description	**Frequency (rad/s)**					
	0.1 (start: pole at -1)	**0.2 (start: pole at -2)**	**0.3 (start: pole at -3)**	**0 (end: pole at -1)**	**20 (end: pole at -2)**	**30 (end: zero at -3)**
Pole at -1	-45	-45	-45	0		
Pole at -2		-45	-45	-45	0	
Zero at -3			45	45	45	0
Total slope (dB/dec)	-45	-90	-45	0	45	0

The resulting component and composite phase plots are shown in the following figure. Since the pole at the origin yields a constant -90° phase shift, the plot begins at -90° and follows the slope portrait just described.



(*a*)



(*b*)

iii. System (iii), with its transfer function:

$$G(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

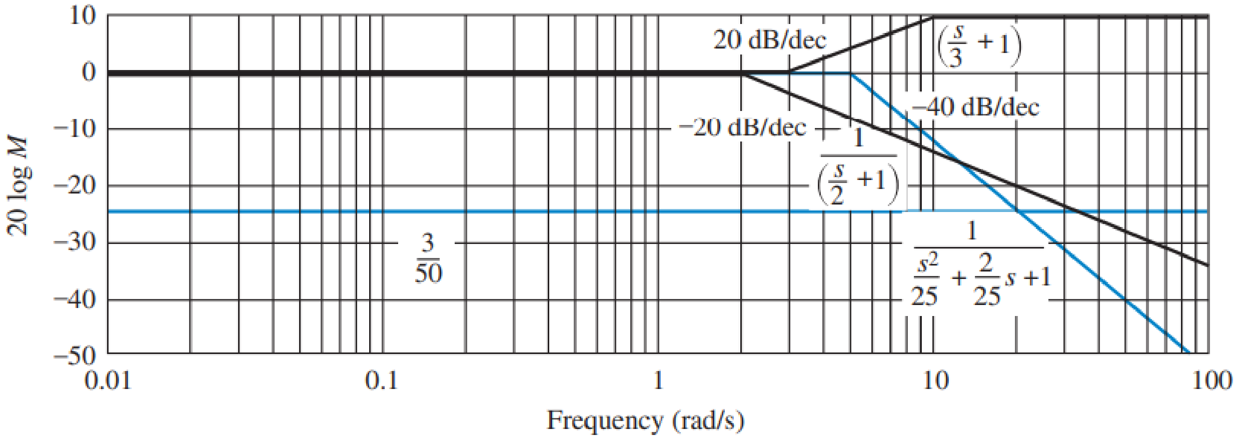
We first convert $G(s)$ to show the normalized components that have unity low-frequency gain. The second-order term is normalized by factoring out ω_n^2, forming:

$$G(s) = s^2 + 2\xi\omega_n s + \omega_n^2 = \frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1$$

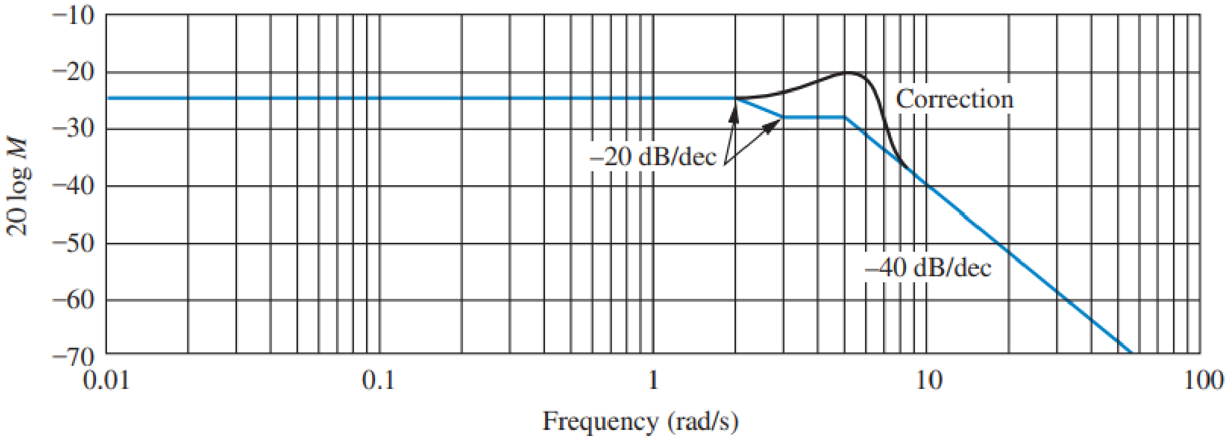
Thus, the system above becomes:

$$G(s) = \frac{3}{(2)(25)}\left[\frac{\frac{s}{3}+1}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)}\right] = \frac{3}{50}\left[\frac{\frac{s}{3}+1}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)}\right]$$

The Bode log-gain diagram is shown in part (b) of the following figure and is the sum of the individual first- and second-order terms of $G(s)$ shown in part (a). We solve this problem by adding the slopes of these component parts, beginning and ending at the appropriate frequencies



(a)



(b)

The results are summarized in the following table. It shows also gain diagram slopes of the given system, which can be used to obtain the slopes.

Description	**Frequency (rad/s)**			
	0.01 (start: plot)	**2 (start: pole at -2)**	**3 (start: zero at -3)**	**5 (start:$\omega_n =$ 5)**
Pole at -2	0	-20	-20	-20
Zero at -3	0	0	20	20
$\omega_n = 5$	0	0	0	-40
Total slope (dB/dec)	0	-20	0	-40

The low-frequency value for $G(s)$, found by letting $s = 0$, is 3/50, or -24.44 dB.

The Bode gain plot starts out at this value and continues until the first break frequency at 2 rad/s. Here the pole at 2 yields a -20 dB/decade slope downward until the next break at 3 rad/s.

The zero at 3 causes an upward slope of +20 dB/decade, which, when added to the previous 20 dB/decade curve, gives a net slope of 0.

At a frequency of 5 rad/s, the second-order term initiates a -40 dB/decade downward slope, which continues to infinity. The correction to the log-gain curve due to the underdamped second-order term can be found by plotting a point $20 \log 2\zeta$ above the asymptotes at the natural frequency.

Since $\zeta = 0.2$ for the second-order term in the denominator of $G(s)$, the correction is 7.96 dB. Points close to the natural frequency can be corrected by taking the values from the curves of the previous figure.

We now turn to the phase plot. The following table that shows the phase diagram slopes of the system is formed to determine the progression of slopes on the phase diagram.

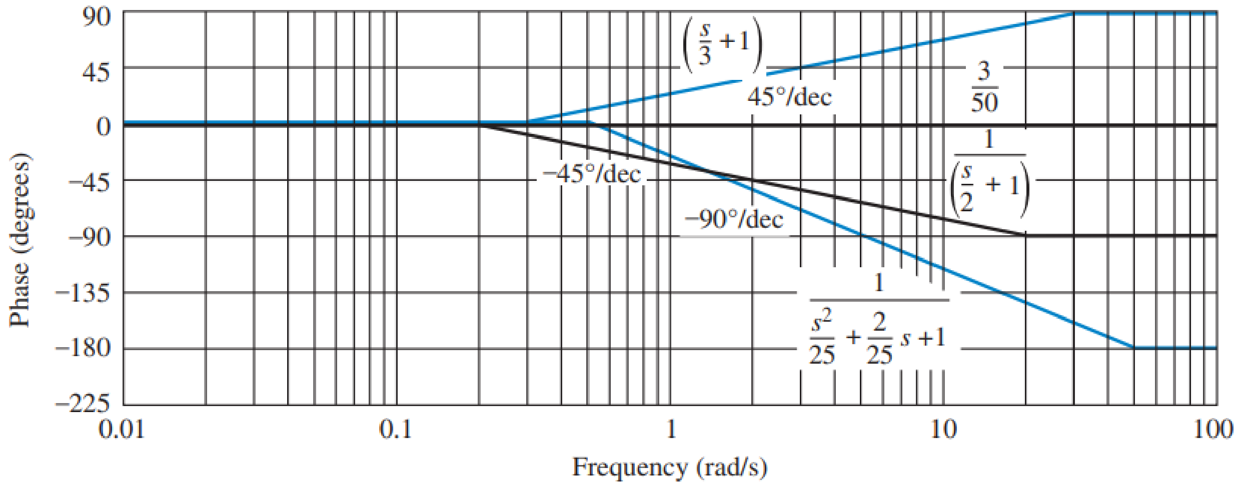
Description	**Frequency (rad/s)**					
	0.2 (start: pole at -2)	**0.3 (start: zero at -3)**	**0.5 (start: ω_n at -5)**	**20 (end: pole at -2)**	**30 (end: zero at -3)**	**50 (end: $\omega_n = 5$)**
Pole at -2	-45	-45	-45	0		
Pole at -3		45	45	45	0	
$\omega_n = 5$			-90	-90	-90	0
Total slope (dB/dec)	-45	0	-90	-45	-90	0

The first-order pole at 2 yields a phase angle that starts at 0° and ends at -90° via a -45°/decade slope starting a decade below its break frequency and ending a decade above its break frequency.

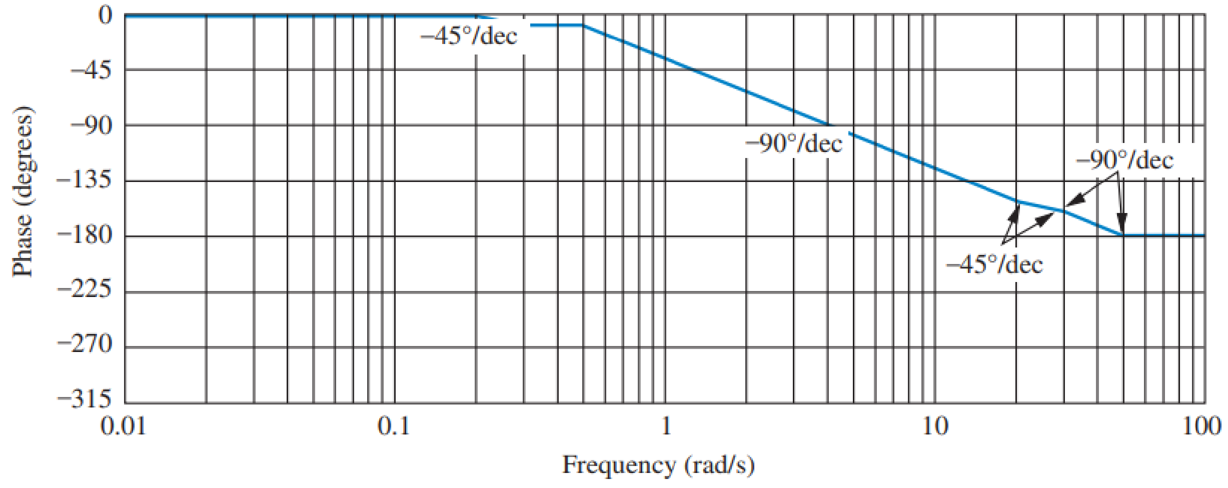
The first-order zero yields a phase angle that starts at 0° and ends at +90° via a +45°/decade slope starting a decade below its break frequency and ending a decade above its break frequency.

The second-order poles yield a phase angle that starts at 0° and ends at -180° via a 90°/decade slope starting a decade below their natural frequency ($\omega_n = 5$) and ending a decade above their natural frequency.

The slopes, shown in the part (a) of the following figure are summed over each frequency range, and the final Bode phase plot is shown in part (b).



(*a*)



(*b*)

b. The description of each system in terms of its frequency response, transient characteristics, stability and steady-state behaviours.

i. System (i), with its transfer function:

$$G(s) = \frac{1}{s+2}$$

Frequency response – The system is a first order system. It has a simple pole at 2 rad/s. The gain at low frequency is -6 dB and its phase shift is 0°. It has a breakpoint at 2 rad/s. At higher frequency, the system settles to a slope of -20 dB/decade and phase shift of -90°.

Transient characteristics – Considering the large gain and phase margins, the system shows an overdamped response when it is subjected to a step input

Stability – Looking into the gain and phase margins of the system that are infinities, the system is stable.

Steady-state behaviours – The system is a type 0 system, so its steady state error is a constant.

ii. System (ii), with its transfer function:

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

Frequency response – Referring to the characteristic equation of the system, the system is a third order system. It has a pole at origin, two simple poles at 1 and 2 rad/s, and a simple zero at 3 rad/s. Notice also parameters K that contributes to the overall gain of the system. The gain at low frequency is a slope at -20 dB/decade and the phase shift is -90°. This indicates that the system is a type 1 system. It has breakpoints at 0, 1, 2, and 3 rad/s. At high frequencies, the gain is a slope at -40 dB/decade and the phase shift is settling at -180°.

Transient characteristics – Looking into its gain and phase margins that are relatively large, the system might have an overdamped response when it is subjected to a step input.

Stability – The gain and phase margins of the system are both positive, so the system is stable.

Steady-state behaviours – Looking into its transfer function, the system is a type 1 system, as it has an integration function in its transfer function. If it is given a step input, the steady-state error of the system is zero.

iii. System (iii), with its transfer function:

$$G(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

Frequency response – Based on its characteristic equation, the system is a third order system. It has a simple pole at 2 rad/s, a pair of complex poles at 5 rad/s, and a simple zero at 3 rad/s. The DC gain at low frequency is -24.44 dB and a phase shift of 0° at 0 rad/s. It has breakpoints at: 2, 3 and 5 rad/s. At high frequencies, the gain is a slope at -40 dB/decade and the phase shift is settling at -180°.

Transient characteristics – Looking into its gain and phase margins that are relatively small and considering the existence of a resonance behaviour around the frequency at the pair complex poles, the system might have an oscillatory transient response when it is subjected to a step input.

Stability – The gain and phase margins of the system are both positive, so the system is stable.

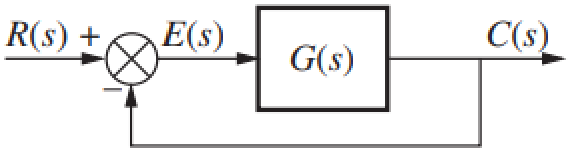
Steady-state behaviours – Looking into its transfer function, the system is a type 0 system, as it does not have an integration function in its transfer function. If it is given a step input, the steady-state error of the system is a constant.

2. Let the transfer function equation of a plant given as follow for the unity feedback system shown in the figure below.

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

a. Use Bode plots to determine the range of K within which the system is stable. [16 marks]

b. If K = 200, find the gain margin and the phase margin. [12 marks]



Solution

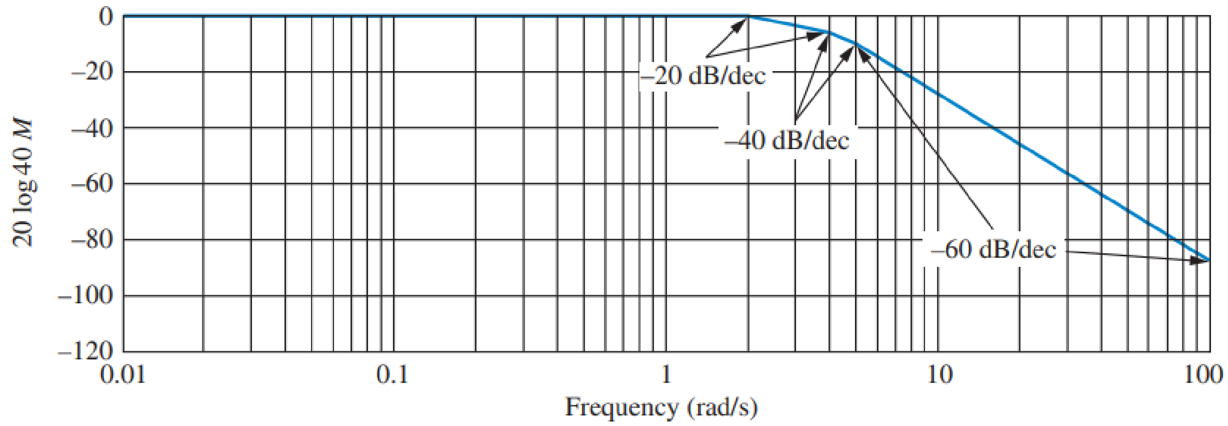
a. Since this system has all its open-loop poles in the left-half plane, the open-loop system is stable. Hence, the closed-loop system will be stable if the frequency response has a gain less than unity when the phase is 180°. Begin by sketching the Bode gain and phase diagrams shown in the following figure.

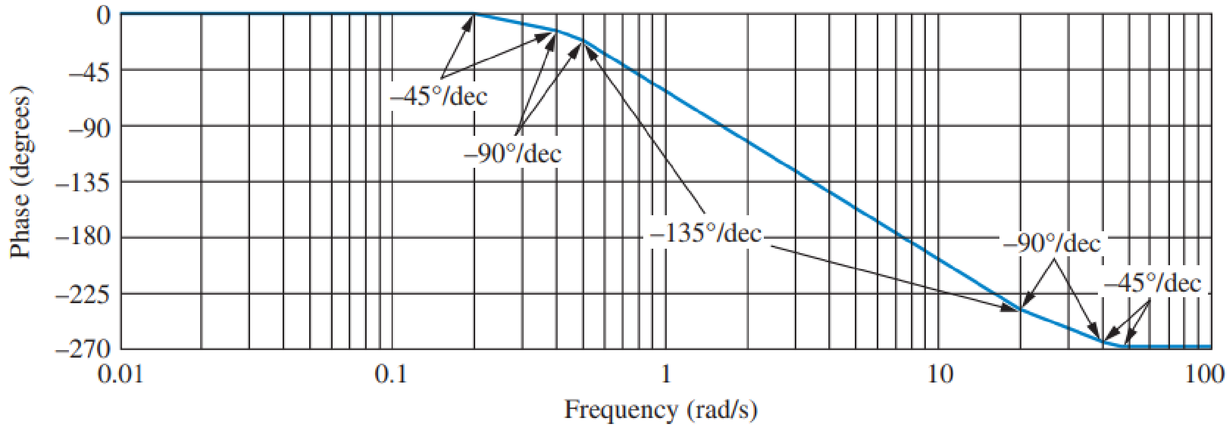
We summed normalized plots of each factor of $G(s)$ to create the Bode plot. We saw that at each break frequency, the slope of the resultant Bode plot changed by an amount equal to the new slope that was added.

In this example, we use this fact to draw the Bode plots faster by avoiding the sketching of the response of each term. The low-frequency gain of $G(s)H(s)$ is found by setting s to zero.

Thus, the Bode gain plot starts at $K/40$. For convenience, let $K = 40$ so that the log-gain plot starts at 0 dB.

At each break frequency: 2, 4, and 5, a 20 dB/decade increase in negative slope is drawn, yielding the log-gain plot shown in the following figure.





The phase diagram begins at 0° until a decade below the first break frequency of 2 rad/s. At 0.2 rad/s the curve decreases at a rate of 45°/decade, decreasing an additional 45°/decade at each subsequent frequency (0.4 and 0.5 rad/s) a decade below each break.

At a decade above each break frequency, the slopes are reduced by 45°/decade at each frequency.

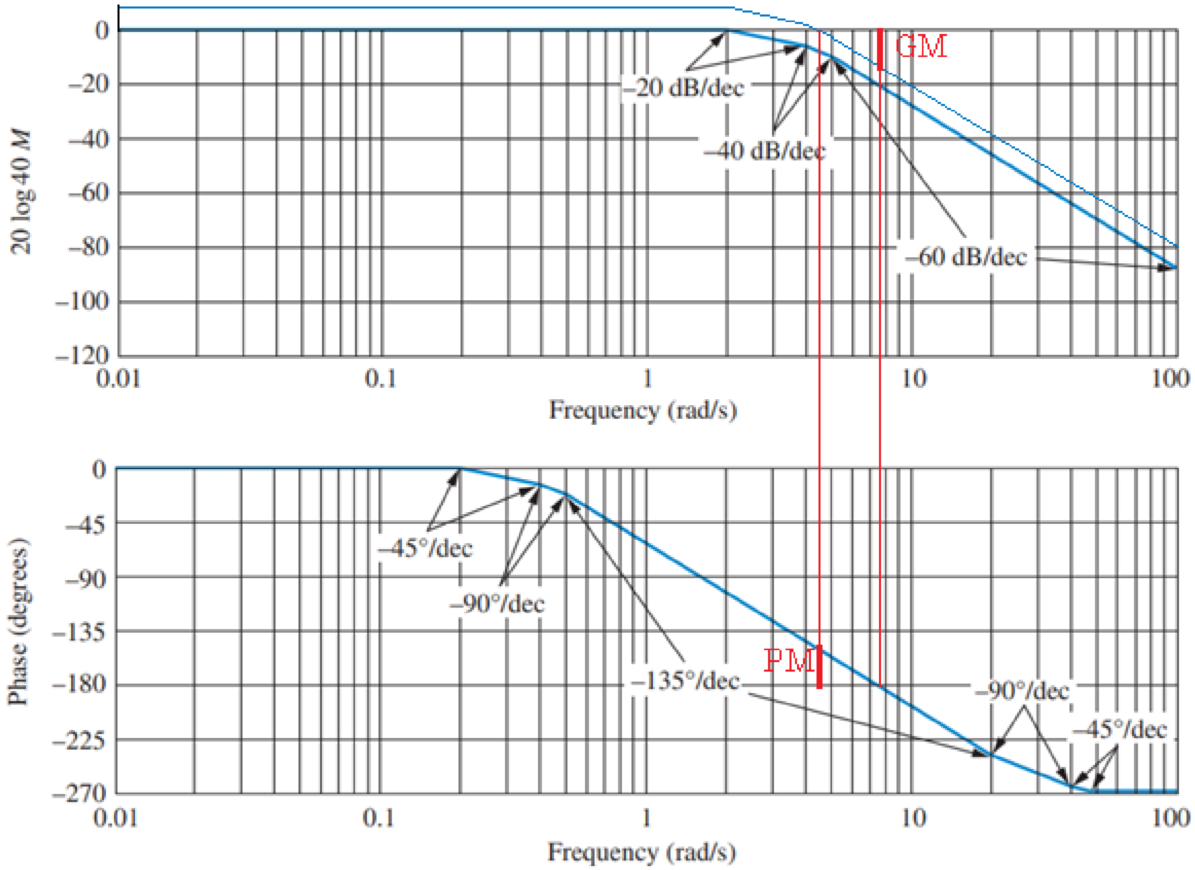
The Nyquist criterion for this example tells us that we want zero encirclements of -1 for stability. Thus, we recognize that the Bode log-gain plot must be less than unity when the Bode phase plot is 180°.

Accordingly, we see that at a frequency of 7 rad/s, when the phase plot is -180°, the gain plot is -20 dB. Therefore, an increase in gain of +20 dB is possible before the system becomes unstable.

Since the gain plot was scaled for a gain of 40, +20 dB (a gain of 10) represents the required increase in gain above 40. Hence, the gain for instability is $40 \times 10 = 400$. The final result is $0 < K < 400$ for stability.

This result, obtained by approximating the frequency response by Bode asymptotes, can be compared to the result obtained from the actual frequency response, which yields a gain of 378 at a frequency of 6.16 rad/s.

c. The Bode plot shown in the figure in part (a) of this question is scaled to a gain of 40. If $K = 200$ (five times as great), the gain plot would be $20\log 5 = 13.98$ dB higher.



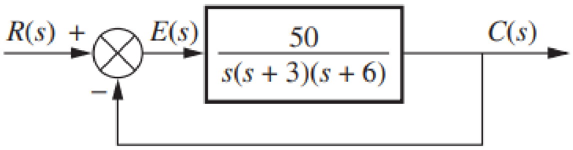
To find the gain margin, look at the phase plot and find the frequency where the phase is 180°.

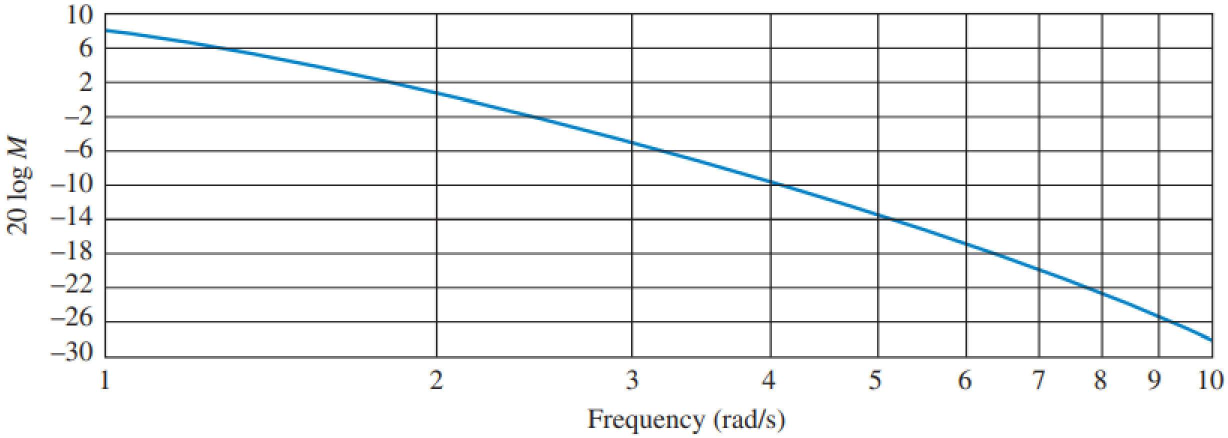
At this frequency, determine from the gain plot how much the gain can be increased before reaching 0 dB. In figure in the part (a) of this question, the phase angle is -180° at approximately 7 rad/s. On the gain plot, the gain is $-20 + 13.98 = 6.02$ dB. Thus, the gain margin is 6.02 dB.

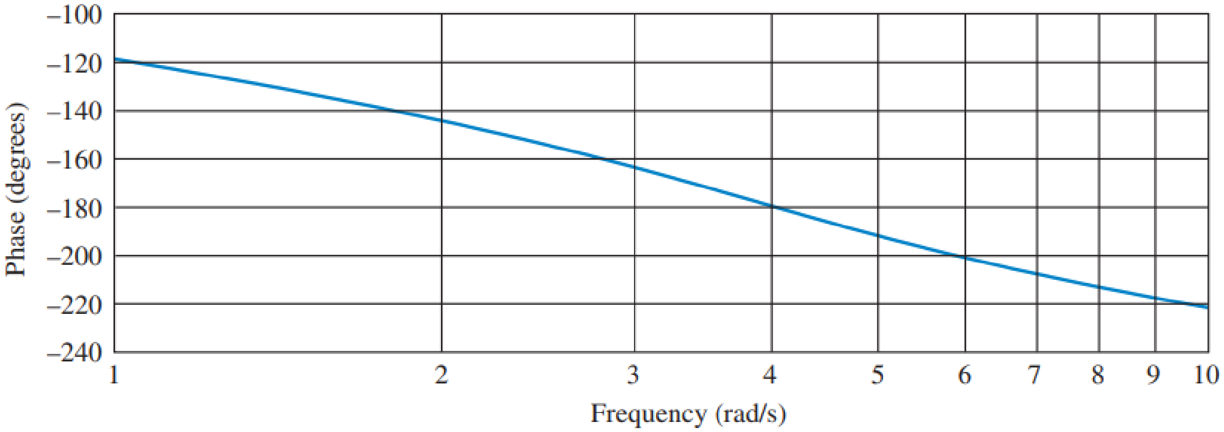
To find the phase margin, we look on the gain plot for the frequency where the gain is 0 dB. At this frequency, we look on the phase plot to find the difference between the phase and -180°. This difference is the phase margin.

Again, remembering that the gain plot of figure in this question part (a) of this question is 13.98 dB lower than the actual plot, the 0 dB crossing (e.g.: -13.98 dB for the normalized plot shown in the part (a) of this question) occurs at 5.5 rad/s. At this frequency, the phase angle is -165°. Thus, the phase margin is: $-165° - (-180°) = 15°$.

3. Given the block diagram of a unity gain feedback closed loop control system and the Bode plots of the system as shown below, estimate the settling time and peak time. [8 marks]





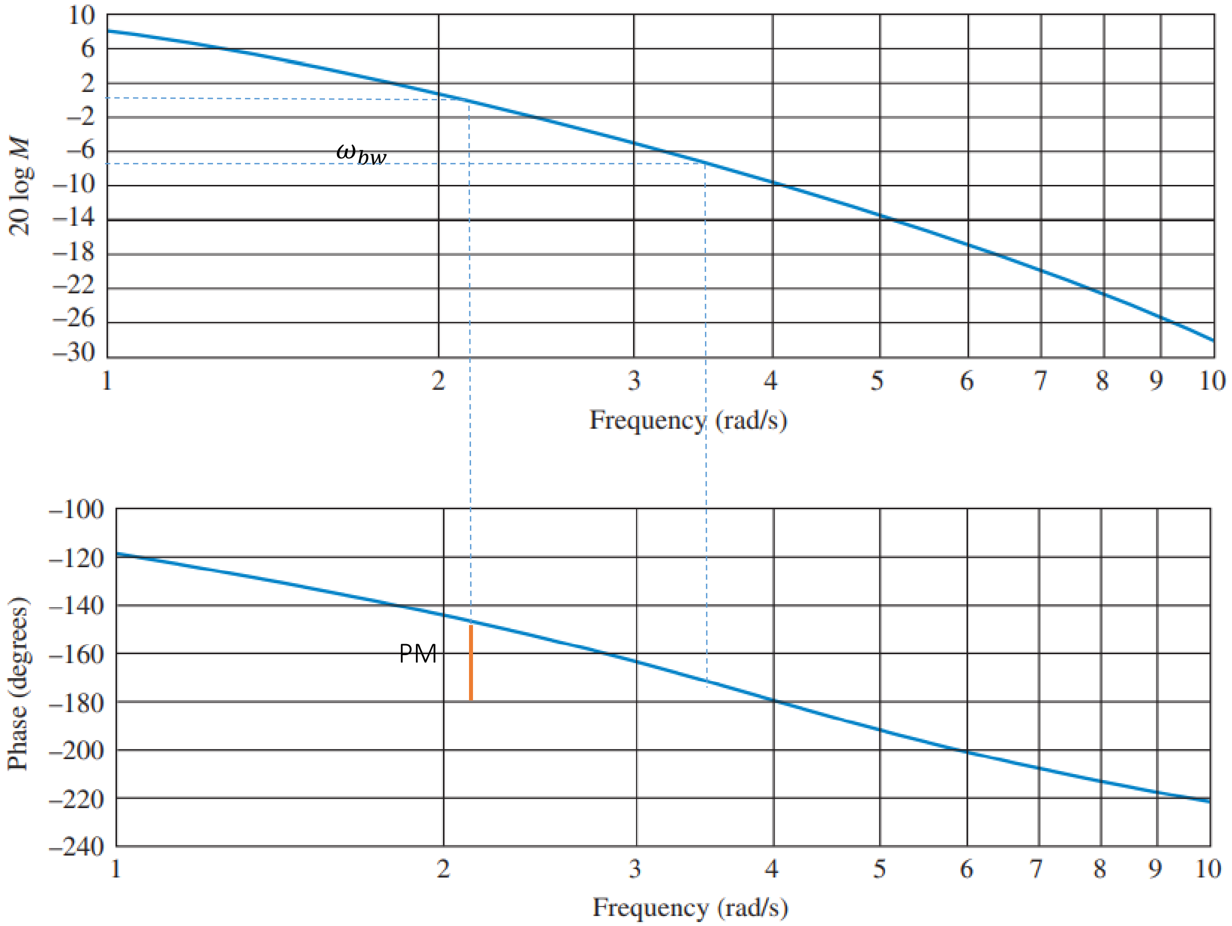


Solution

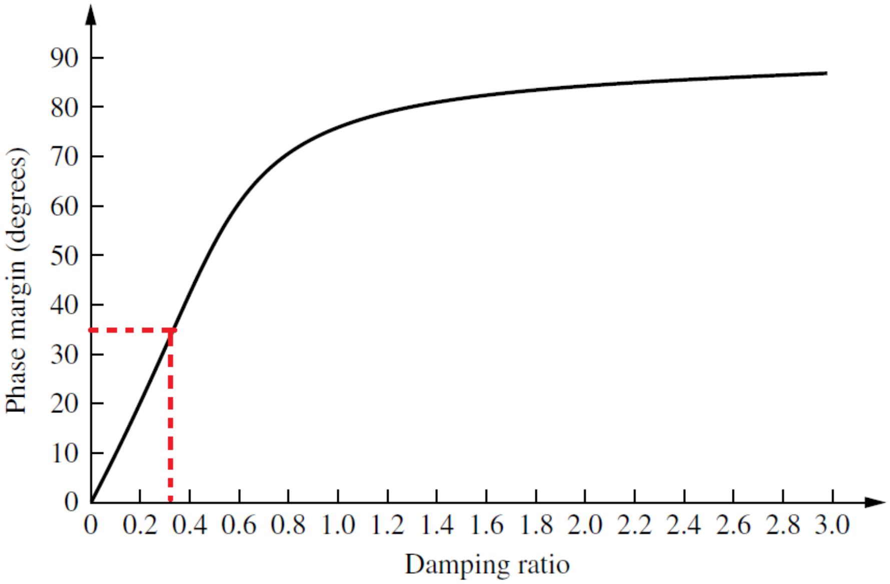
Using the Bode plots in the figure above, we estimate the closed-loop bandwidth by finding the frequency where the open-loop gain response is in the range of -6 to -7.5 dB if the phase response is in the range of -135° to -225°.

Since the Bode plots in the figure above shows -6 to -7.5 dB at approximately 3.7 rad/s with a phase response in the stated region, $\omega_{BW} \cong$ 3.7 rad/s.

Next, find ζ via the phase margin. From the Bode plots shown in the figure given above, the phase margin is found by first finding the frequency at which the gain plot is 0 dB. At this frequency, 2.2 rad/s, the phase is about -145°. Hence, the phase margin is approximately $-145° - (-180°) = 35°$.



Using the figure given below, ζ = 0.32 when the phase margin is 35°. Alternatively, the damping ratio is about 0.35 from the graph given below.



Finally, knowing $\omega_n = 4/T_s\zeta$ using the following equations,

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Hence

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Since $\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}}$ the above equation becomes:

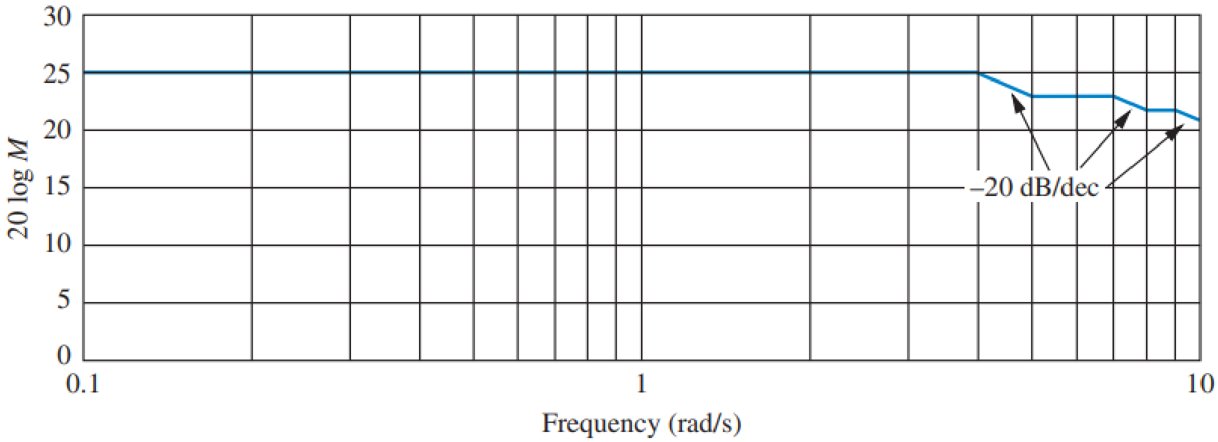
$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Hence

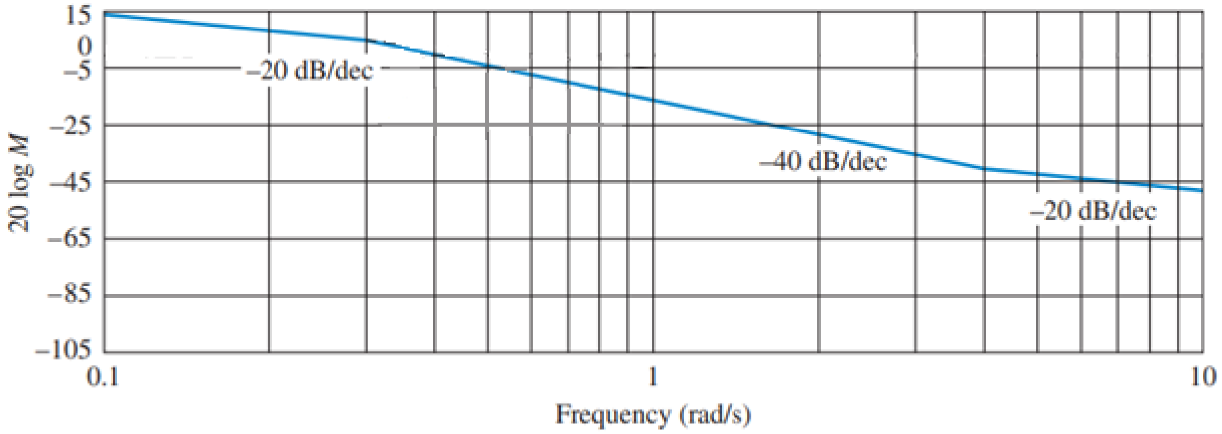
$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

With the values of ω_{BW} and ζ just found, $T_s = 4.86$ seconds and $T_p = 1.29$ seconds, checking the analysis with a computer simulation shows $T_s = 5.5$ seconds, and $T_p = 1.43$ seconds.

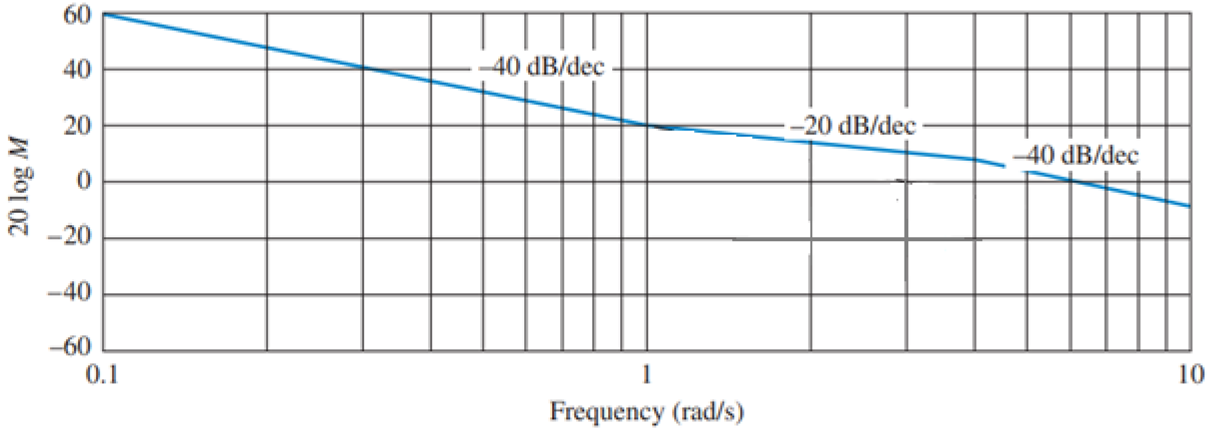
4. Referring to each un-normalized and unscaled Bode log-gain plot of three control systems as shown in the figures below, find the system types and the value of the appropriate static error constants.
[12 marks]



(*a*)



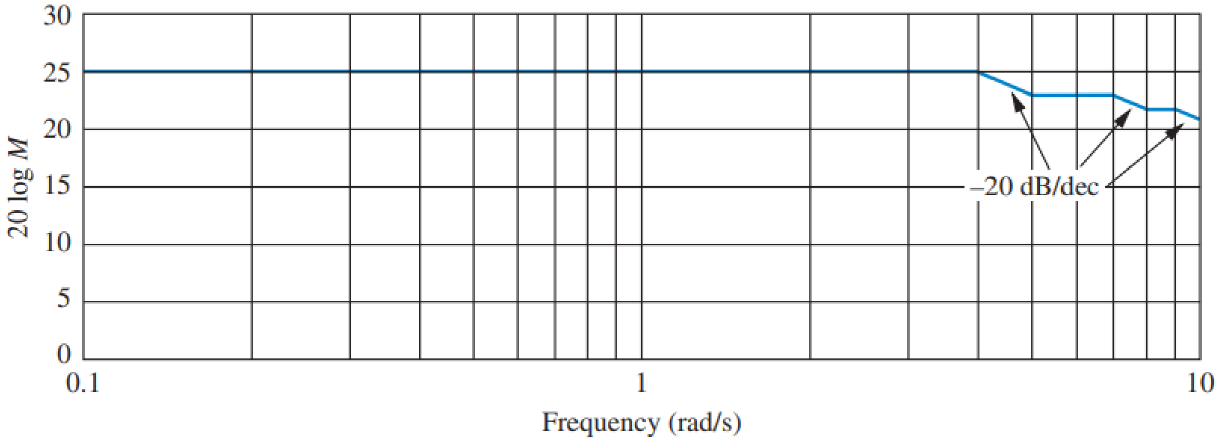
(*b*)



(*c*)

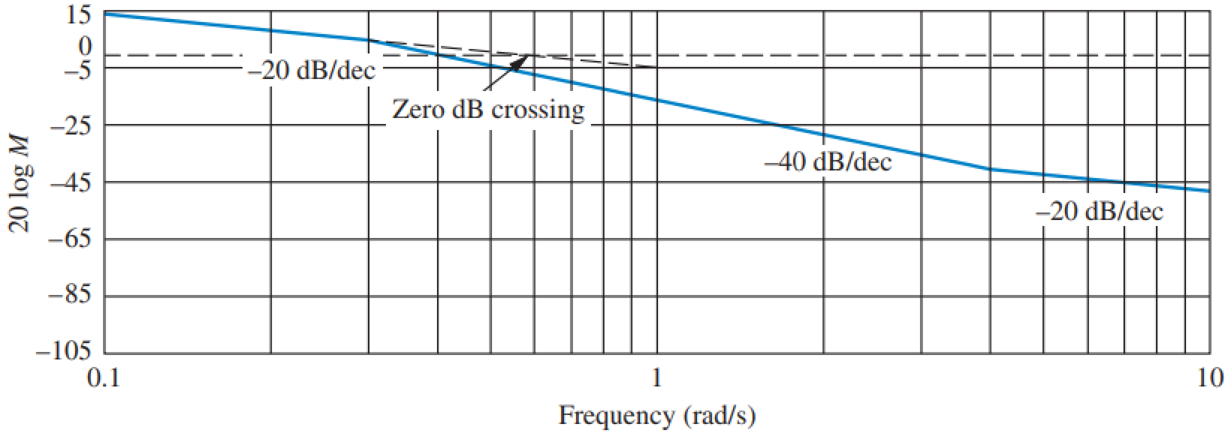
Solution

Figure part (a) is a Type 0 system, since the initial slope is zero. The value of K_p is given by the low-frequency asymptote value. Thus, $20 \log K_p = 25$, or K_p = 17.78.



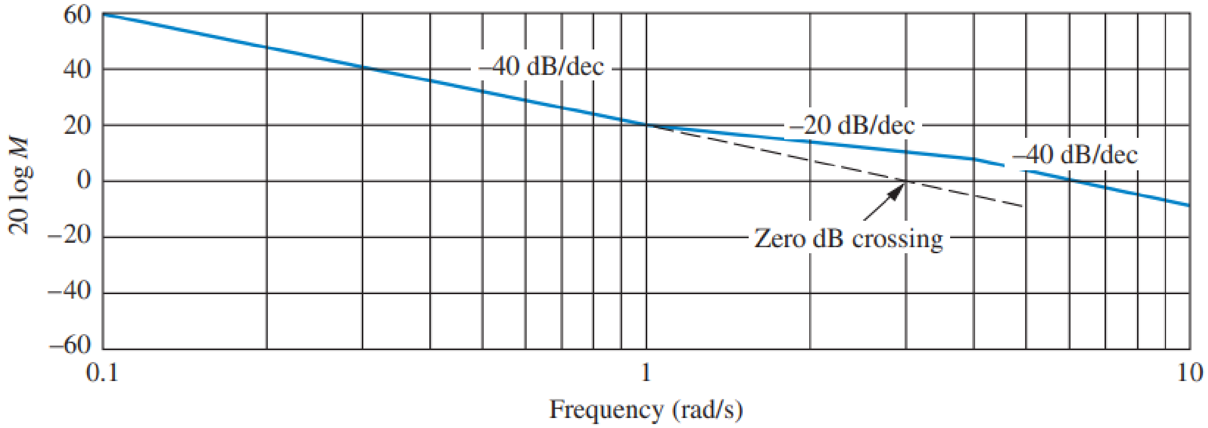
(*a*)

Figure part (b) is a Type 1 system, since the initial slope is -20 dB/decade. The value of K_v is the value of the frequency that the initial slope intersects at the zero dB crossing of the frequency axis. Hence, from the graph the frequency,ω is 0.55 rad/s, so K_v = 0.55.



(b)

Figure part (c) is a Type 2 system, since the initial slope is -40 dB/decade. The value of $\sqrt{K_a}$ is the value of the frequency that the initial slope intersects at the zero dB crossing of the frequency axis. Hence, from the graph the frequency,ω is 3 rad/s, so $K_a = (3)^2 = 9$.



(c)