LECTURE 3: PROPAGATION OF ERROR

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Introduction

- Any measuring procedure contains error.
- Thus, measured values generally differ somewhat from the true values that are being measured.
- The errors in the measurements produce errors in calculated values (like the mean).

Definition: When error in measurement produces error in calculated values, we say that error is **propagated** from the measurements to the calculated value.

Section 3.1: Measurement Error

- A geologist weighs a rock on a scale and gets the following measurements:

  251.3   252.5   250.8   251.1   250.4

- These measurements differ from one another, and it is unlikely that any of them is equal to the true mass of the rock.
- The **error** in the measured value is the difference between a measured value and the true value.

Parts of Error

- We think of the error of the measurement as being composed of two parts:
  - **Systematic error (bias)**
  - **Random error**
- Bias is the part of the error that is the same for every measurement. For example, a scale that always gives you a reading that is too low.
- Random error is error that varies from measurement to measurement and averages out to zero in the long run.

Parts of Error

- Any measurement can be considered to be the sum of the true value plus contributions from each of the components of error:

Measured value = true value + bias + random error

Two Aspects of the Measuring Process

- We are interested in **accuracy**.
  - Accuracy is determined by bias.
  - The smaller the bias, the more accurate the measuring process.
  - If the bias is zero, the measuring process is said to be **unbiased**.

Two Aspects of the Measuring Process

- We are also interested in **precision**.
  - Precision refers to the degree to which repeated measurements of the same quantity tend to agree with each other.
  - If repeated measurements come out nearly the same every time, the precision is high.

More on Error

- A measured value is a random variable with mean μ and standard deviation σ.
- The bias in the measuring process is the difference between the mean measurement and the true value:

$$\textbf{Bias} = \mu - \textbf{true value}$$

- The uncertainty in the measuring process is the standard deviation σ.
- The smaller the bias, the more accurate the measuring process.
- The smaller the uncertainty, the more precise the measuring process.

Example 1

A laboratory sample of gas is known to have a carbon monoxide (CO) concentration of 50 parts per million (ppm). A spectrophotometer is used to take five independent measurements of this concentration. The five measurements, in ppm, are 51, 47, 53, 53, and 48. Estimate the bias and the uncertainty in a spectrophotometer measurement.

Error Continued

Let $X_1, \ldots, X_n$ be independent measurements, all made by the same process on the same quantity. The sample standard deviation s can be used to estimate the uncertainty.

- Estimates of uncertainty are often crude, especially when based on small samples.
- If the true value is know, the sample mean, $\bar{X}$, can be used to estimate the bias:

$$\text{Bias} \approx \bar{X} - \text{true value}$$

- If the true value is unknown, the bias cannot be estimated from repeated measurements.

Example 1 cont.

The spectrophotometer has been recalibrated, so we may assume that the bias is negligible. The spectrophotometer is now used to measure the CO concentration in another gas sample. The measurement is 55.1 ppm. How should this measurement be expressed?

Section 3.2: Linear Combinations of Measurements

- If X is a measurement and c is a constant, then

$$\sigma_{cX} = |c|\sigma_X$$

- If X_1, …, X_n are independent measurements and c_1, …, c_n are constants, then

$$\sigma_{c_1X_1+\cdots+c_nX_n} = \sqrt{c_1^2\sigma_{X_1}^2 + \cdots + c_n^2\sigma_{X_n}^2}$$

Example 2

A surveyor is measuring the perimeter of a rectangular lot. He measures two adjacent sides to be 50.11 ± 0.05m and 75.12 ± 0.08m. These measurements are independent. Estimate the perimeter of the lot and find the uncertainty in the estimate.

Repeated Measurements

If $X_1, \ldots, X_n$ are n *independent* measurements, each with mean μ and standard deviation σ, then the sample mean, $\bar{X}$, is a measurement with mean

$$\mu_{\bar{X}} = \mu$$

and with uncertainty $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}}$

Example 3

The length of a component is to be measured by a process whose uncertainty is 0.05 cm. If 25 independent measurements are made and the average of these is used to estimate the length, what will the uncertainty be? How much more precise is the average of 25 measurements than a single measurement?

Repeated Measurements with Differing Uncertainties

If X and Y are *independent* measurements of the same quantity, with uncertainties σ_X and σ_Y, respectively, then the weighted average of X and Y with the smallest uncertainty is given by

$c_{\text{best}} X + (1 - c_{\text{best}}) Y$, where

$$c_{\text{best}} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \qquad 1 - c_{\text{best}} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2}.$$

Linear Combinations of Dependent Measurements

If $X_1, \ldots, X_n$ are measurements and $c_1, \ldots, c_n$ are constants, then

$$\sigma_{c_1X_1+\cdots+c_nX_n} \leq |c_1|\sigma_{X_1} + \cdots + |c_n|\sigma_{X_n}$$

Example 4

A surveyor is measuring the perimeter of a rectangular lot. He measures two adjacent sides to be 50.11 ± 0.05 m and 75.21 ± 0.08 m. These measurements are not necessarily independent. Find a conservative estimate of the uncertainty in the perimeter of the lot.

Section 3.3: Uncertainties for Functions of One Measurement

- If X is a measurement whose uncertainty σ_X is small, and if U is a function of X, then

$$\sigma_U \approx \left| \frac{dU}{dX} \right| \sigma_X \qquad (3.10)$$

  In practice, we evaluate the derivative dU/dX at the observed measurement X.

- This is the **propagation of error formula**.

Relative Uncertainties for Functions of One Measurement

- If U is a measurement whose true value is μ_U, and whose uncertainty is σ_U, the relative uncertainty in U is the quantity σ_U / μ_U.

- The relative uncertainty is a unitless quantity. It is frequently expressed as a percent. In practice μ_U is unknown, so if the bias is negligible, we estimate the relative uncertainty with σ_U / U.

Example 5

Question: The radius R of a sphere is measured to be 3.00 ± 0.001cm. Estimate the volume of the sphere and its uncertainty.

Answer: The volume of the sphere, V, is given by $V = 4\pi R^3/3$. The estimate of V is $4\pi(3)^3/3 = 113.097$cm^3. Now, $\sigma_R = 0.001$cm and $dV/dR = 4\pi R^2 = 36\pi$ cm^2. We can now find the uncertainty in V:

$$\sigma_V = \left|\frac{dV}{dR}\right|\sigma_R = (36\pi \text{ cm}^2)(0.001 \text{ cm}) = 1.056 \text{ cm}^3.$$

We estimate the volume of the sphere to be 113.097 ± 1.056 cm^3.

Approximating Relative Uncertainty

- There are two methods for approximating the relative uncertainty σ_U / U in a function $U = U(X)$:
  - Compute σ_U using Equation (3.10), and then divide by U.
  - Compute $\ln U$ and use Equation (3.10) to find $\sigma_{\ln U}$, which is equal to σ_U / U .

Recall: Equation (3.10) is $$\sigma_U \approx \left| \frac{dU}{dX} \right| \sigma_X$$

Relative uncertainty is a number without units. It is frequently expressed as a percent.

Example 6

Find the relative uncertainty from the example of a sphere.

Answer: We found the volume of the sphere to be 113.097 ± 1.056 cm^3. The relative uncertainty is $\sigma_V / V = 1.056/113.097 = 0.00934$. We can express the volume as $V = 113.097$ cm$^3 \pm 9.34\%$.

Section 3.4: Uncertainties for Functions of Several Measurements

If $X_1, \ldots, X_n$ are *independent* measurements whose uncertainties $\sigma_{X_1}, \sigma_{X_2}, \ldots, \sigma_{X_n}$ are small, and if $U = U(X_1, \ldots, X_n)$ is a function of $(X_1, \ldots, X_n)$, then

$$\sigma_U \approx \sqrt{\left(\frac{\partial U}{\partial X_1}\right)^2 \sigma_{X_1}^2 + \left(\frac{\partial U}{\partial X_2}\right)^2 \sigma_{X_2}^2 + \cdots + \left(\frac{\partial U}{\partial X_n}\right)^2 \sigma_{X_n}^2}$$

In practice, we evaluate the partial derivatives at the point $(X_1, \ldots, X_n)$.

Uncertainties for Functions of Dependent Measurements

If $X_1, \ldots, X_n$ are *independent* measurements whose uncertainties $\sigma_{X_1}, \sigma_{X_2}, \ldots, \sigma_{X_n}$ are small, and if $U = U(X_1, \ldots, X_n)$ is a function of $(X_1, \ldots, X_n)$, then a conservative estimate of σ_U is given by

$$\sigma_U \leq \left|\frac{\partial U}{\partial X_1}\right|\sigma_{X_1} + \left|\frac{\partial U}{\partial X_2}\right|\sigma_{X_2} + \cdots + \left|\frac{\partial U}{\partial X_n}\right|\sigma_{X_n}$$

Uncertainties for Functions of Dependent Measurements

In practice, we evaluate the partial derivatives at the point (X_1, …, X_n).

This inequality is valid in almost all practical situations; in principle it can fail if some of the second partial derivatives of U are quite large.

Example 7

Two perpendicular sides of a rectangle are measured to be $X = 2.0 \pm 0.1$ cm and $Y = 3.2 \pm 0.2$ cm. Find the absolute uncertainty in the area $A = XY$.

Answer: First, we need the partial derivatives:

$$\frac{\partial A}{\partial X} = Y = 3.2 \text{ and } \frac{\partial A}{\partial Y} = X = 2.0$$

,

so the absolute uncertainty is

$$\sigma_A = \sqrt{3.2^2(0.01) + 2.0^2(0.04)} = 0.5122.$$

Summary

- We discussed measurement error.
- Then we talked about different contributions to measurement error.
- We looked at linear combinations of measurements (independent and dependent).
- We considered repeated measurements with differing uncertainties.
- The last topic was uncertainties for functions of several measurements.