TE WHARE WĀNANGA O TE ŪPOKO O TE IKA A MĀUI
VICTORIA
UNIVERSITY OF WELLINGTON

EXAMINATIONS — 2013

END-OF-YEAR

SWEN 224
Formal Foundations of Programming

Time Allowed: 3 Hours

Instructions:

- Answer all **four** questions.
- The exam will be marked out of one hundred and eighty (180).
- Calculators ARE NOT ALLOWED.
- Non-electronic foreign language dictionaries are allowed.
- No other reference material is allowed.

Question 1. Assertions [45 marks]

Let A be an array of integers indexed from 1 to $|A|$ and B be an array of integers indexed from 1 to $|B|$. Let x, y and z be integers.

(a) Consider the following assertion:

$$x < y \land z = y$$

(i) [1 mark] State in words what the assertion means.

For each of the following, say whether the assertion is true or false:

(ii) [1 mark] $x = 2, y = 3, z = 3$

(iii) [1 mark] $x = 3, y = 3, z = 3$

(b) Consider the following assertion:

$$\forall j \bullet 1 \leq j \leq |A| \rightarrow A[j] > 50$$

(i) [2 marks] State in words what the assertion means.

For each of the following arrays, say whether the assertion is true or false:

(ii) [1 mark] $A = ()$

(iii) [1 mark] $A = (101, 102, 49)$

(c) Consider the following assertion

$$\left(\forall j \bullet 1 \leq j < |A| \rightarrow A[j] \leq A[j+1]\right) \land \left(\Sigma_{i=1}^{i=|A|} A[i] \leq 0\right)$$

(i) [4 marks] State in words what the assertion means.

For each of the following arrays, say whether the assertion is true or false.

(ii) [1 mark] $A = (-10, 0, 1, 3, 3)$

(iii) [1 mark] $A = (-10, 9, 0)$

(iv) [1 mark] $A = ()$

(v) [1 mark] $A = (-1, 0)$

(d) Write an assertion formalising each of the following statements:

(i) [4 marks] x is greater than y and less then z.

(ii) [4 marks] A contains at least one occurrence of d.

(iii) [4 marks] If A contains z then it dose not contains y.

(iv) [4 marks] The sum of the elements in A is greater than the largest element in B.

(v) [4 marks] No element in A is in B.

(e) Program contracts.

(i) [5 marks] Write a formal contract, a pre-condition and post-condition, for the following program.

The program takes two arrays, A and B, containing the lengths and widths respectivly of a sequence of rectangles. So $A[i]$ and $B[i]$ are the length and width of rectangle i (the ith rectangle). The program outputs an array, C, containing the area of the rectangles. Note the area of a rectangle is defined by $area = length \times width$.

(ii) [5 marks] Write a formal contract, a pre-condition and post-condition, for the following program.

The program inputs, P, an array of pairs. When i is an index of P then the first element of pair $P[i]$ is written as $first(P[i])$ and is a student name. The second element of $P[i]$ is written as $second(P[i])$ and is the integer grade for the student. The program outputs:

1. SB - the set of students what have grade "B", that is their numeric grade is greater than or equal to 70 and less than 75.
2. CB - the number of students that have grade "B".

Question 2. Verifying Programs [45 marks]

Each question has a program and a post-condition. Use Hoare Logic to compute the missing conditions, simplifying your answers where appropriate.

Variables $x, y, z, w \ldots$ are integers and A is an array of integers.

(a) [6 marks] What are the conditions: **P, P1** and **P2**?

P $\ldots$

$y := 3x;$

P1 $\ldots$

$y := y + 2x;$

P2 $\ldots$

$x := y + 2;$

Q $\{x \times y > 20\}$

(b) [6 marks] What are the conditions: **P, P1, P2** and **P3**?

P $\ldots$

P1 $\ldots$

$x := x - 1;$

Q $\{10 > x + y\}$

P2 $\ldots$

$y := -y;$

P3 $\ldots$

$x := 12$

Q $\{10 > x + y\}$

If $(0 \leq x \wedge 0 \leq y)$ **then** $x := x - 1$**; else** $y := -y; x := 12$**; fi**

Q $\{10 > x + y\}$

(c) [6 marks] What are the conditions: **P, P1, P2** and **P3**, and the statement **S1**?

P . . .

> **P1** . . .
>
> $y := y + 1;$
>
> **P2** . . .
>
> $x := x - y;$
>
> **Q** $\{x < 10\}$

> **P3** . . .
>
> **S1** . . .
>
> **Q** $\{x < 10\}$

If $(x < 2)$ **then** $y := x + 1; x := x - y;$ **fi**

Q $\{x < 10\}$

(d) [6 marks] What are the conditions: **P, P1, P2** and **P3**?

P . . .

$w := y + 1;$

P1 . . .

$x := y;$

P2 . . .

$A[x] := 4;$

P3 . . .

$z := A[w];$

Q $\{z > 10\}$

(e) [3 marks] For a while program with guard **B** and loop body **S**:

```
while B do { S }
```

state the three loop conditions using pre-condition **P** and post-condition **Q**.

(f) Consider the following program:

```
int i := 0;
int x := 0;
while (i < y) then {
   x := x + b;
   i := i + 1;
}
```

(i) [3 marks] Given the loop pre-condition $\mathbf{P} = \{i = 0 \land x = 0 \land 0 < y \land b \times y \leq 16\}$, loop post-condition $\mathbf{Q} = \{x \leq 16\}$ and the program above define:

(I) the loop invariant **L**

(II) the variant **V** (a value that decreases and is bounded below)

that will satisfy the three loop conditions.

(ii) [3 marks] Prove your loop invariant satisfies the three loop conditions for the given program, pre-condition and post-condition.

(g) The program below takes two arrays A and B of the same size and computes s the difference between the sum of the elements in A and the sum of the elements in B.

```
int i := 0;
int s := 0;
while i < |A| then {
    i := i + 1;
    s := s + A[i] - B[i];
}
```

(i) [6 marks] Define the following conditions:

(I) the post-condition to the while loop, **Q**

(II) the pre-condition to the while loop, **P**

(III) the loop invariant **L**

(ii) [6 marks] Prove your loop invariant, pre-condition and post-condition satisfy the three loop conditions for the given program.

Question 3. Regular Languages [45 marks]

(a) [8 marks] Write a *Regular Expression* to describe each of the following languages:

(i) All strings over $\{a, b\}$ which start and end with an a and do not contain two consecutive b's.

(ii) All strings over $\{a, b, c\}$ containing exactly one a or exactly one b (or both). For example, this language includes the strings a, $cbaccaaa$, $cbbba$ and $ccacccbc$, but not $aacc$, $cccbbccc$ or $aabbb$.

(b) [6 marks] Consider the language consisting of all strings over $\{a, b, c\}$ in which no a is followed immediately by a b, no b is followed immediately by a c and no c is followed immediately by an a (i.e. no string contains ab, bc or ca as a substring).

(i) Draw a transition diagram for a DFA that accepts this language.

(ii) Give a formal definition for your DFA in part **(i)**.

(c) [6 marks] Draw a transition diagram for an NFA$_\epsilon$ that accepts the language defined by the regular expression $a(ab|b^*(a|c))^*a$.

(d) Consider the NFA $M = (Q, q_I, A, N, F)$, where:

$$\begin{aligned}
Q &= \{1, 2, 3, 4, 5, 6\} \\
q_I &= 1 \\
A &= \{\text{a, b, c}\} \\
N &= \{(1, a, 1), (1, a, 2), (1, a, 3), (2, a, 2), (2, a, 4), (3, a, 3), (3, b, 3), (3, b, 5), \\
&\quad (4, a, 4), (4, b, 4), (4, c, 6), (5, a, 5), (5, b, 5), (5, c, 5), (5, b, 6)\} \\
F &= \{5,6\}
\end{aligned}$$

(i) [4 marks] Draw a transition diagram for M.

(ii) [5 marks] Show the sequence of configurations that M passes through in accepting the input $aababc$.

Note that you should show *all* states that M may be in after accepting part of the input.

(iii) [8 marks] Draw a transition diagram for the DFA obtained by applying the *subset construction* to M. Show the correspondence between states of the DFA and those of the NFA. You only need to show reachable states.

(e) [8 marks] Given a DFA $M = (Q, q_I, A, N, F)$ which accepts a language L, show how to construct a DFA, $M' = (Q', q'_I, A', N', F')$, which accepts the language L' consisting of all *prefixes* of strings in L, i.e. $L' = \{\alpha \in A^* \mid \exists \beta \bullet \alpha\beta \in L\}$.

Give a brief argument to show that the resulting DFA does in fact accept the required language.

Question 4. Context-Free Languages [45 marks]

(a) Suppose you want to use a simple language to describe the arrangement of shapes on a screen, where shapes can be nested within other shapes. Each shape is either a square or a circle, and is described by the position of its centre (a pair of natural numbers) and, optionally, a list of shapes contained within it. For example, the first example below describes a circle containing a square and another circle; the second example describes a square containing the circle from the first example, along with another circle:

- `circle 80 60 [square 70 50, circle 90 90]`
- `square 50 50 [circle 80 60 [square 70 50, circle 90 90], circle 20 20]`

(i) [10 marks] Write a *Context Free Grammar* to describe this language.

You are not required to give a full formal definition of this grammar — just write the list of rules. In writing this grammar, you should treat `number` as a terminal symbol and not define it further.

(ii) [5 marks] Draw a parse tree for the first example above using your grammar and assuming that 80, 60, etc. are instances of `number`.

(iii) [10 marks] Is your grammar LL(1)? If not, turn it into one that is.

Prove that the resulting grammar is LL(1).

(b) The following is part of a grammar for simple set expressions, where C stands for set comprehension and is not specified here:

$$E \to C \mid E \cup E \mid E \cap E \mid E - E \mid (\ E\ )$$

(i) [6 marks] Explain, using an example, why this grammar is ambiguous.

(ii) [8 marks] Write an equivalent, unambiguous grammar, treating $\cup$, $\cap$ and $-$ as *left-associative*, and giving $\cup$ and $-$ the same precedence, and $\cap$ higher precedence than $\cup$ and $-$.

(c) [6 marks] Let $G = (V_N, V_T, S, P)$ be a Context Free Grammar which defines a language L.

Show how you can extend G to obtain a grammar G' which defines a language $L' = \{(\alpha) \mid \alpha \in L\}$, i.e. the language consisting of strings from L enclosed in a pair of brackets.

Give a brief argument to show that your grammar defines exactly the required language.

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