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Technically, functions are just another type of relation, although, psychologically we think of them in a different way. We begin by giving some definitions that work for any relation.

Suppose R is a relation from A to B.

The **domain** of R is the set

$$\{a \in A : aRb \; for some \; b \in B\}.$$

For example, if A is the set of women, B is the set of people, and R_1 is defined by xR_1y if x is the mother of y, then the domain of R_1 is the set of all mothers.

The **image** or **range** of R is the set

$$\{b \in B : aRb \; for some \; a \in A\}.$$

For example, consider the relation $R_2 \subseteq \mathbb{R} \times \mathbb{R}$ defined by $(x, y) \in \mathbb{R}$ if $y = x^2$. Then the range of R_2 is the set of non-negative real numbers.

The relation R_2 above is an example of a "function". Specifically a relation f from a set A to a set B is a **function** if

(i) the domain of f is A;
(ii) if $(a, b) \in f$ and $(b, c) \in f$, then $b = c$.

- Condition (i) says that each $a \in A$ is f–related to *at least* one $b \in B$.
- Condition (ii) says that each $a \in A$ is f–related to *at most* one $b \in B$.
- Altogether, every $a \in A$ is f–related to *exactly* one $b \in B$.

Notation: Instead of saying $(a, b) \in f$ we almost always say $f(a) = b$, and instead of saying $f \subseteq A \times B$, we usually say $f : A \to B$.

It's hard to believe, but the above simple definition encapsulates some 2000 years of thinking about the idea of what a function really is. Loosely speaking, a function f describes a process, whereby, when a is the input and is processed by f, the output is b.

In this processing language, we see that

- everything gets processed, and
- after processing we get a single output.

We are surrounded by functions in so many ways that it is hard to know where to begin. As I type this, I see that the symbols S that appear on my computer screen are a function of the keys K that I press. Here we have a function $f : K \to S$. (Hmmmm, as I think about it, I'm not sure that this is really a function. Why?)

In practice, we are most often given functions as rules, ie, given $x \in A$, the rule tells you how to get $f(x) \in B$. Thus

$$f = \{(x, y) : y = x^2 + 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

is a function described by a rule. We would more commonly describe it by

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2 + 1.$$

If $f : A \to B$ is a function, then B is the **codomain** of f. Note that B is not necessarily the same as the image or range of the function. For example, let W be the set of all women and P be the set of all people. Define $f : P \to W$ by letting $f(x)$ be the mother of x. I sincerely hope that this is a function, ie, every person has exactly one mother (but with modern genetics who knows?) Now the *range* of this function is the set of all mothers, while the *codomain* is the set of all women—not the same.

Or consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ discussed above. Here the codomain is $\mathbb{R}$, while the range is $\{x \in \mathbb{R} : x \geq 1\}$. (Why?)

It is important to recognise when a relation is *not* a function. Here are two basic examples.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then f is not a function because $f(0)$ is not defined. (Why?)

How could we turn this relation into a function?

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. This is not a function for two reasons. First,
   - f is not defined for negative numbers.
   - f is not *single valued*, for example, both 2 and -2 are square roots of 4.

A function f is **onto** or **surjective** if the codomain of f is equal to its range; in other words, $f : A \to B$ is surjective if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

How do we decide that f is not surjective. We need only a single instance where the property fails. In other words, we need a single *counterexample.*

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n$. Then there is no n such that $f(n) = 1$; thus f is not surjective.

On the other hand, if we want to show that f is surjective we need a proof. A typical proof goes like this.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - 2x$. Is f onto? Choose $y \in \mathbb{R}$. If $y = f(x)$, then we have $y = 1 - 2x$, and, solving for x gives $x = \frac{y-1}{2}$. Clearly, we can always find an x for every y. Thus f is onto.

Crucial Observation: If we want to show that a property does not hold in general, all we need is a *single counterexample*. If we want to show that a property does hold, we need a *proof.*

It is quite common with functions to have the situation where $a \neq b$, but $f(a) = f(b)$. For example, with $f(x) = x^2$, we see that $4 \neq -4$, but $f(4) = f(-4) = 16$. If this does not happen, then the function is one-to-one or injective. More precisely, a function $f : A \to B$ is **one-to-one** or **injective** if, for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

- It is often useful to use the contrapositive, ie, if $x \neq y$, then $f(x) \neq f(y)$, ie, different things go to different places.
- We have multiple terminology: one-to-one or injective, and onto or surjective. Sadly, both sets of terminology are used throughout mathematics. Such is life.

To show that a function is *not* one-to-one, all we need is a single counterexample, just as we showed with $f(x) = x^2$, on the other hand, to show that a function is injective, we need a proof.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 1$. Assume that $f(x_1) = f(x_2)$, then $2x_1 - 1 = 2x_2 - 1$. This implies that $2x_1 = 2x_2$ and hence $x_1 = x - 2$. This *proves* that f is one-to-one.

The Rolls-Royces of functions are both one-to-one and onto. Such functions are called **bijections** or **one-to-one correspondences**. A function $f : A \to B$ is a bijection if for each $a \in A$, there is exactly one $b \in B$ such that $b = f(a)$, and for each $b \in B$, there is exactly one $a \in A$ such that $f(a) = b$.

Here is a somewhat whimsical example. Consider the set of all women and the set of all men. Define the relation f by letting $f(w)$ be the husband of w. This is *not* a function. Why?

- Some women are not married. Thus there exists a woman w for which $f(w)$ is not defined.
- There are societies in which women can have more than one husband. Thus f is not single valued.

Here we can do a trick and massage f into a function by restricting the domain. Let's restrict ourselves to the set of married women in societies in which women can have only one husband. Now f is indeed a function.

Is f onto. Well, I believe that there exist unmarried men. So f is not onto. Let's do the trick again, and restrict our codomain to the set of married men.

Now f is an onto function, but f is not one-to-one, because there exist a number of societies in which men can have more than one wife.

The moral of our story is:

Whether or not a relation is a function, or is onto etc depends not just on the rule we use to define the function, but also on the choice of domain and range.

Inverses

Because a function is a relation, we can always define its inverse as follows: $f^{-1}(w) = z$ if and only if $f(z) = w$. However, typically, the inverse will not even be a function. But for bijections we have:

Theorem 9.1. *Let $f : A \to B$ be a bijection. Then $f^{-1} : B \to A$ is a bijection.*

Proof. This proof is best done using the ordered pair notation, that is, instead of saying $f(a) = b$, we say $(a, b) \in f$. We have to show four things.

(1) Since f is onto, we see that for each $b \in B$, there is an a such that $(a, b) \in f$. Thus, for each $b \in B$, there is an a such that $(b, a) \in f^{-1}$. This shows that f^{-1} *is defined for all elements of its domain.*
(2) Since, f is one-to-one, if $(a_1, b) \in f$ and $(a_2, b) \in f$, then $a_1 = a_2$. Thus, if $(b, a_1) \in f^{-1}$, and $(b, a_2) \in f^{-1}$, then $a_1 = a_2$. This shows that f^{-1} is *single valued*.
    Combining 1 and 2, we now know that f^{-1} is at least a function.
(3) Since f is a function, if $a \in A$, there is a $b \in B$ such that $(a, b) \in f$. This shows that, if $a \in A$, there is a $b \in B$ such that $(b, a) \in f^{-1}$. This shows that f^{-1} is *onto*.
(4) Again, since f is a function (and therefore single-valued) if $(a, b_1) \in f$, and $(a, b_2) \in f$, then $b_1 = b_2$. Thus, if $(b_1, a) \in f^{-1}$, and $(b_2, a) \in f^{-1}$, then $b_1 = b_2$, so that f^{-1} is *one-to-one*.

Putting it all together we see that f^{-1} is indeed a bijection. □

Composing functions. Given $f : A \to B$ and $g : B \to C$ then the composition of f and g denoted by $g \circ f : A \to C$ is defined by $g \circ f(a) = g(f(a))$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$, and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 + x + 1$. Then we have

$$g \circ f(x) = g(f(x)) = g(2x+1) = (2x+1)^2 + (2x+1) + 1 = 4x^2 + 6x + 3.$$

For the composition of f and g to be a function, f has to be onto. (Why?)

Theorem 9.2. *If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is also a bijection.*