Logic Intro

Logic...

- gives precise meaning to mathematical statements
- is the basis of mathematical and automated reasoning
- helps in designing:
  - computing machines (example: circuits)
  - system specifications
  - programming languages
  - artificial intelligence

Furthermore,

- we need to understand what constitutes a correct argument to:
  - verify computer programs
  - ensure algorithms are correct
  - automate proofs

Propositions

- Logic is the study of principles of valid reasoning
- Propositional logic is a simple language for working with mathematical statements
- A proposition or statement is a sentence that is either true or false
- We often use P or Q to denote a proposition

Propositions

- We write ‘1’ or ‘T’ for a true statement and ‘0’ or ‘F’ for a false one
- Every statement is assigned one of these values (since by definition, every statement is either true or false)
- The problem is figuring out which statements are true and which are false

Propositions

- The following sentences *are* propositions:

Example

- $2 + 2 = 3$.
- Wellington is the southernmost capital city.

- The following sentences *are not* propositions:

Example

- What is your name?
- Give me your money.
- This sentence is a lie.
- $x + 2 = 6$.

Inference

- We are often interested in inferring one statement from others.
- For example, from the statements:
    - all dogs are animals and
    - Fido is a dog,

  we can correctly infer that Fido is an animal. This appears to be correct or valid reasoning.
- However, given the statements
    - all dogs are animals and
    - Bucky is an animal,

  it would be invalid reasoning to deduce that Bucky is necessarily a dog. Why?

Combining Propositions

- A compound proposition is built from other propositions using *connectives*.
- "I will read my notes before every lecture and I will do my homework" is built from the primitive propositions
  - "I will read my notes before every lecture"
  - "I will do my homework".
- In the above compound statement "and" functions as a logical connective between the primitive statements "I will read my notes before every lecture" and "I will do my homework".

Conjunction (AND)

- $P \wedge Q$ (i.e. 'P and Q')
- True only when both P and Q are true

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

- A truth table tells us how the truth of a compound proposition relates to the truth of its primitive components (True=1; False = 0).

Disjunction (OR)

- $P \lor Q$ (i.e. ‘P or Q’)
- True when at least one of P and Q is true

P	Q	$P \lor Q$
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive-Or (XOR)

- 'P xor Q'
- True when **exactly** one of P and Q is true
- "Either P or Q is true"

P	Q	P xor Q
0	0	0
0	1	1
1	0	1
1	1	0

- In English the word "or" has two different meanings. If someone said that they want "fish or steak for dinner", you would not expect them to eat both. This is the **exclusive** use of "or"
- On the other hand, the statement "running or swimming is good for you" is true, even though doing both is also good for you. This is the **inclusive** use of "or", also known as disjunction
- In logic we use the inclusive sense of "or" by default

Negation (NOT)

- $\neg P$ (i.e. 'not P' or 'it is not the case that P')
- True when P is false

P	$\neg P$
0	1
1	0

A tautology is a compound proposition that is true, regardless of the truth value of its components

A contradiction is a compound proposition that is false regardless of the truth value of its components

A contingent proposition is one that is neither a tautology nor a contradiction. Most propositions are contingent

Practice Questions

1. Use a truth table to prove that $P \vee \neg P$ is a tautology
2. Use a truth table to prove that $P \wedge \neg P$ is a contradiction

Implication (implies)

- $P \to Q$ ('P implies Q')
- Corresponds (roughly) to the English statement: if P then Q

P	Q	$P \to Q$
0	0	1
0	1	1
1	0	0
1	1	1

Implication (implies)

- The truth-table for implication has some counter-intuitive properties, called false premises
- People are often surprised that $P \to Q$ is true when P is false and Q is true. For example,
    - If I swim regularly, then I will get fit.
- It is quite possible that you do not swim regularly but get fit for a different reason (e.g. running regularly).
- True or False: "If Wellington is the largest city in China then Toronto is the capital of France"?
- The above example shows an even weirder property of implication: $P \to Q$ is true when both P and Q are false. In computer science, this is known as "*Garbage in, garbage out*"

Practice Questions

- What is the truth table for $P \wedge (P \rightarrow Q)$?
- Use a truth table to prove that:

Theorem

The compound propositions $P \rightarrow Q$ and $\neg P \vee Q$ are logically equivalent.

Converse

- $Q \to P$

P	Q	$Q \to P$
0	0	1
0	1	0
1	0	1
1	1	1

- In English:
  - $P \to Q$ can be written as "Q if P" or "if P then Q"
  - $Q \to P$: can be written as "Q only if P"
- Note that $P \to Q$ and $Q \to P$ have very different meanings. "All engineering students are undergrads" is not the same as "all undergrads are engineering students"

Contrapositive

- $\neg Q \to \neg P$

P	Q	$\neg Q \to \neg P$
0	0	1
0	1	1
1	0	0
1	1	1

Practice Question

Prove the following

Theorem

The compound propositions $P \to Q$ and $\neg Q \to \neg P$, i.e. implication and contrapositive, are logically equivalent.

An English example is:

- If it is sunny, then A goes for a run
- If A doesn't go for a run, then it is not sunny

Equivalence

- $P \leftrightarrow Q$ ('P if and only if Q', 'P iff Q')
- Corresponds to: P and Q are logically equivalent
- Also know as biconditionality

P	Q	$P \leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

- We also use $P \equiv Q$ to denote logical equivalence.

Practice Questions

- Prove that $P \leftrightarrow Q$ and $(P \rightarrow Q) \wedge (Q \rightarrow P)$ are logically equivalent.
- Write down the "win condition" for tic-tac-toe as a compound proposition.

Laws of Logic

- Show $\neg(\neg P) \equiv P$.
- This is known as the law of the excluded middle: in (classical) logic there is no "grey area" in between true and false.
- If something is not false, then it is true; and if something is not true, then it is false.
- Show that $\neg(P \wedge Q)$ is not logically equivalent to $(\neg P \wedge \neg Q)$

Laws of Logic

Prove De Morgan's laws:

Definition

De Morgan's Laws

- $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
- $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

List of laws:

1. Double negation: $P \equiv \neg\neg P$
2. De Morgan's laws:
   $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
   $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$
3. $P \rightarrow Q \equiv \neg P \vee Q$
4. Commutative laws:
   $P \wedge Q \equiv Q \wedge P$
   $P \vee Q \equiv Q \vee P$
5. Idempotent laws:
   $P \wedge P \equiv P$
   $P \vee P \equiv P$
6. Distributive laws:
   $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

List of laws:

7. Associative laws:
   $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
   $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
8. Contrapositive: $(P \to Q) \equiv (\neg Q \to \neg P)$
9. Tautology: if $\mathbb{T}$ is a tautology, then
   $P \vee \mathbb{T} \equiv \mathbb{T}$
   $P \wedge \mathbb{T} \equiv P$
10. Contradiction: if $\mathbb{F}$ is a contradiction, then
    $P \vee \mathbb{F} \equiv P$
    $P \wedge \mathbb{F} \equiv \mathbb{F}$

Practice Question

- Use the laws above to show that

  $$(P \to Q) \vee \neg Q$$

  is a tautology.
  "If I run then I'll get fit" **or** "I won't get fit".
- Show that
  $$((P \to Q) \to Q) \equiv (P \vee Q)$$
- Some people prefer the following notation:

  $$\neg P \longrightarrow \overline{P}$$
  $$P \wedge Q \longrightarrow P \cdot Q$$
  $$P \vee Q \longrightarrow P + Q$$

List of laws:

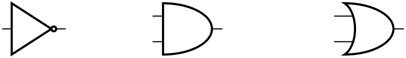
1. Double negation: $P \equiv \overline{\overline{P}}$
2. De Morgan's laws:
   $\overline{P \cdot Q} \equiv \overline{P} + \overline{Q}$
   $\overline{P + Q} \equiv \overline{P} \cdot \overline{Q}$
3. $P \rightarrow Q \equiv \overline{P} + Q$
4. Commutative laws:
   $P \cdot Q \equiv Q \cdot P$
   $P + Q \equiv Q + P$
5. Idempotent laws:
   $P + P \equiv P$
   $P \cdot P \equiv P$
6. Distributive laws:
   $P + (Q \cdot R) \equiv (P + Q) \cdot (P + R)$
   $P \cdot (Q + R) \equiv (P \cdot Q) + (P \cdot R)$
7. etc.

Practice Question

Simplify the expression

$$PQR + PQ\overline{R}$$

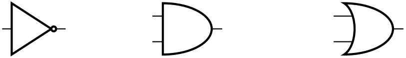
Application: Logic Circuits



Inverter AND gate OR gate

Example: Given p, q and r, design a circuit that outputs $(p \wedge \neg q) \vee \neg r$.

Application: Logic Circuits



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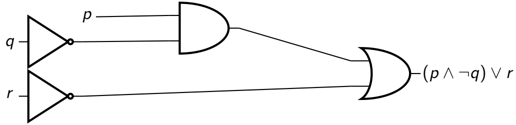

Binary arithmetic

- Construct a circuit that (correctly!) adds two single-digit binary numbers
- Construct a circuit that adds three single-digit binary numbers
- Construct a circuit that adds two double-digit binary numbers