Logical quantifiers

Predicates and Quantifiers

Statements like

- *"all countries have capital cities"* and
- *"some cities have more than a million people"*

are more complicated than the examples we looked at previously, which were more **specific**: e.g.

- *"NZ has a capital city"* and
- *"Wellington has more than a million people."*

Definition: *Predicate*

A predicate $P(x)$ is a sentence that contains a variable x and becomes a proposition whenever a specific value is substituted for the variable.

- $P(x)$ is "x is capital of New Zealand"
- $P(\text{Wellington})$ = True. $P(\text{Timaru})$=False.

Note that *before* we substituted Wellington/Timaru in for x, the predicate $P(x)$ was *neither* true *nor* false.

Quantifiers

Universal quantifier: Denoted by $\forall$, read as "for all"
$\forall x\, P(x)$ means "every x satisfies $P(x)$" or
"for all x, $P(x)$ is true"

Existential quantifier: Denoted by $\exists$, read as "there exists"
$\exists x\, P(x)$ means "at least one x satisfies $P(x)$" or
"there exists an x, such that $P(x)$ is true"

- The *domain* of a predicate variable is the set of all values that may be substituted in place of the variable.
- For example let $P(x) =$"x can navigate by echolocation", and let $M = \{\text{mammals}\}$ and $H = \{\text{humans}\}$ be two domains. Then the statement

$$\exists x \in M\ P(x) \text{ is true, whereas}$$
$$\exists x \in H\ P(x) \text{ is presumably false.}$$

- Often, we won't bother specify the domain.

Quantifiers

Example

1. "There exists an 18-year old student in ENGR123"
   Let $A(x)$ mean that "x is 18 years old" and
   $E(x)$ mean that "x is enrolled in ENGR123"
   The statement becomes
   $$\exists x \left(A(x) \land E(x) \right)$$
2. "All students at Vic study engineering"
   Let $V(x)$ mean "x is a student at Vic" and
   $G(x)$ mean that "x studies engineering"
   The statement is
   $$\forall x \left(V(x) \implies G(x) \right)$$

   Q: what if there are no students at Vic?

Practice Questions

1. You work at a hotel in Fiji and need to find a customer a room for a week. Let S be the set of rooms in the hotel and D be the set of days of the relevant week. Finally, let predicate $F(r, d)$ mean "room r is free on day d".
   Consider the statements

   $$(\forall d \in D)\ (\exists r \in R)\ F(r, d) \text{ and}$$
   $$(\exists r \in R)\ (\forall d \in D)\ F(r, d).$$

   Which one will make the customer happy and why?
   The *order* in which quantifiers occur can make a huge difference.
2. What do the following statements mean, and which of them is true?
   - $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x < y)$
   - $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(x < y)$

Back to Graphs:

- A walk on a graph is an alternating sequences of vertices and edges

$$v_1 e_1 v_2 e_2 \cdots e_{n-1} v_n$$

  such that each edge e_i is incident with both vertices v_i and v_{i+1}
- A path is a walk where no vertex appears more than once
- A cycle is a walk where only the *first* and *last* vertices are equal
- A graph is *connected* if there is a path connecting any two points.
- Define "connected" using predicate logic.

Truth and Falsity of Universal Statements

- A universal statement, i.e. a statement of the form $\forall x \in D,\ P(x)$ is
  - true iff $P(x)$ is true for every $x \in D$.
  - false iff there is (at least) one $x \in D$ for which $P(x)$ is false.
    Such an x is called a **counterexample.**
- All swans are white.
- $S(x) =$"x is a swan"; $W(x)$="x is white".

$$\forall x,\ S(x) \rightarrow W(x).$$

- An **existential statement** is:
  - true iff $P(x)$ is true for (at least) one $x \in D$.
  - false iff there is $P(x)$ is false for every $x \in D$.

Negation and Quantifiers

- The negation of $\forall x,\ P(x)$ is

$$\neg\Big(\forall x,\ P(x)\Big) \equiv \Big(\exists x,\ \neg P(x)\Big).$$

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Practice Questions

Negate the following:

- All software engineers code in binary.
- No software engineer codes in binary.
- Some engineers code in binary.
- Some engineers don't code in binary.
- If a program contains $> 100,000$ lines, then it has a bug.
- If a program is written by Jeff, then it has no bugs.

Taking the last two together, what can you infer?

Question:

Why didn't mathematicians define

$$\neg\Big(\forall x,\ P(x)\Big) \equiv \Big(\forall x,\ \neg P(x)\Big)?$$

e.g. What is wrong with

$$\neg\text{“All swans are white”} \equiv \text{“no swans are white”}?$$

Back to the hotel in Fiji

- Negate:
  1. $(\forall d \in D)\ (\exists r \in R)\ F(r,d)$
  2. $(\exists r \in R)\ (\forall d \in D)\ F(r,d)$
- Negate:
  1. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x < y)$
  2. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(x < y)$

Generalised DeMorgan's Laws

Theorem

If $P(x)$ is a predicate with universal set U, then

$$\forall x \in U : \neg P(x) \Leftrightarrow \neg(\exists x \in U : P(x)),$$

$$\exists x \in U : \neg P(x) \Leftrightarrow \neg(\forall x \in U : P(x)).$$

- $\forall x : (P(x) \wedge Q(x)) \Leftrightarrow \forall x : P(x) \wedge \forall x : Q(x)$.
- $\exists x : (P(x) \vee Q(x)) \Leftrightarrow \exists x : P(x) \vee \exists x : Q(x)$.

Predicates in Several Variables

A predicate can be of the form $P(x_1, x_2, \ldots, x_n)$, where each x_i comes from some universal set U_i. So the overall universal set is

$$U1 \times U2 \times \cdots \times U_n = \{(x_1, x_2, \ldots, x_n) : x_i \in U_i \forall i = 1, 2, \ldots, n\}.$$

Example

Let $Q(x, y)$ denote the statement $x = y + 3$. Then the truth values of

- $Q(1, 2)$ is false since $1 \neq 5$
- $Q(4, 1)$ is true since $4 = 1 + 3$.

Quantification of Predicates with several variables

Example

1. $\exists x : (x + y = 0)$
   x is bound and y is free.
2. $\exists x : (P(x) \wedge Q(x)) \vee \forall x : R(x)$

A predicate is a proposition when **all** its variables must be bound.

Multiple Variables and Mixing Quantifiers

Let $A(x)$: "x is an astronaut," $P(y)$: "y is a planet", and $V(x, y)$: "x will travel to y."

- Every astronaut will travel to at least one planet.
  $\forall x : [A(x) \rightarrow (\exists y : (P(y) \rightarrow V(x, y)))]$
- Some astronauts will travel to every planet.
  $\exists x : [A(x) \wedge (\forall y : (P(y) \rightarrow V(x, y)))]$
- Other astronauts will travel to no planets.
  $\exists x : [A(x) \wedge (\forall y : (P(y) \rightarrow \neg V(x, y)))]$
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  Every planet will be visited by some astronaut.
- $\forall x : [P(x) \rightarrow \exists y : (A(y) \wedge \neg V(y, x))]$.
  No planet will be visited by all astronauts.

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