2017

Student ID: ....

Question 5. DISCRETE RANDOM VARIABLES [15 marks]

(a) [4 marks] The discrete random variable, X, takes on the values in the set $\{-2,-1,0,1,2\}$ with the following probabilities: $P(X=-2)=\frac{2}{10}$, $P(X=-1)=\frac{1}{10}$, $P(X=0)=\frac{4}{10}$, $P(X=1)=\frac{2}{10}$, $P(X=2)=\frac{1}{10}$.

(i) Find the probability that $|X|$ is greater than 1.

$$P(|X|>1) = P(X>1 \text{ or } X<-1) = P(X=-2)+P(X=2)$$
$$= 3/10$$

(ii) Find E(X)

$$E(X) = -2(2/10) + -1(\tfrac{1}{10}) + 0(4/10) + 1(2/10) + 2(1/10)$$
$$= -1/10$$

(iii) Find Var(X)

$$E(X^2) = 4(2/10) + 1(1/10) + 0(4/10) + 1(2/10) + 4(1/10)$$
$$= 15/10$$

$$Var(X) = \frac{15}{10} - \left(\frac{-1}{10}\right)^2 = \frac{15}{10} - \left(\frac{1}{10}\right)^2$$

(b) [3 marks]

If X is a binomial random variable with parameters $n=10$ and $p=0.2$, what is the mean and variance of Y and Z where $Y=-2X+5$ and $Z=2(X+1)$?

$E(X) = 2 \quad Var(X) = 1.6$

$E(Y) = -2(2)+5 = 1 \qquad E(Z) = 2(2+1) = 6$

$Var(Y) = 4(1.6) = 6.4 \qquad Var(Z) = 4(1.6) = 6.4$

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(c) [3 marks]

Let the random variable, X, be a Binomial random variable with parameter $n = 8$ and $p = 0.3$. What is the probability $P(X = 5)$?

$$\binom{8}{5} 0.3^5 \, 0.7^3$$

(d) [3 marks] Let the random variable, X, count the number of the arrival of requests at a server from 9am to 10am. What are the assumptions needed for X to be well modelled as a Poisson random variable? Are these assumptions likely to hold?

arrivals should be
- single
- independent
- constant rate

seem reasonable unless requests trigger other requests or if there is a reason why 9am is different to 10am.

(e) [2 marks] The geometric random variable X has the probability mass function (pmf):

$$p(X = k) = (1 - p)^k p, \qquad k = 0, 1, \ldots,$$

Given the fact that $S = \sum_{k=1}^{\infty}(1 - p)^k = \frac{1 - p}{p}$ and the derivative of S with respect to p is nothing else but the sum $S' = -\sum_{k=1}^{\infty} k(1 - p)^{k-1}$, verify that the expected value of X is

$$\mathrm{E}(X) = \frac{1 - p}{p}.$$

You have another way of doing this and this is slightly different to the way we have defined geometric

Question 6. CONTINUOUS RANDOM VARIABLES [15 marks]

(a) [8 marks] Let the random variable X have the probability density function (pdf), $f(x)$, where:

$$f(x) = \begin{cases} Ax^3, & 1 \leq x \leq 3, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Answer the following questions:

(i) Find A.

$$\int_1^3 Ax^3\,dx = \left[\frac{Ax^4}{4}\right]_1^3 = \frac{A}{4}(3^4 - 1^4) = 20A = 1$$

$$\Rightarrow A = \frac{1}{20}$$

(ii) Find $P(X < 2)$.

$$\int_1^2 \frac{1}{20}x^3\,dx = \left[\frac{x^4}{80}\right]_1^2 = \frac{16}{80} - \frac{1}{80} = \frac{15}{80}$$

(iii) Find $E(X)$.

$$\int_1^3 x \times \frac{x^3}{20}\,dx = \int_1^3 \frac{x^4}{20}\,dx = \left.\frac{x^5}{100}\right]_1^3 = \frac{3^5 - 1}{100}$$

$$= \frac{242}{100}$$

(b) [4 marks]

Let the random variable X have the cumulative distribution function (cdf), $F(x)$, where:

$$F(x) = \begin{cases} 1 - \exp(-2x), & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (2)$$

Answer the following questions:

(i) Find the probability density function of X.

$$f(x) = F'(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(ii) Find $P(1 < X \leq 2)$.

$$F(2) - F(1) = (1 - e^{-4}) - (1 - e^{-2})$$
$$= e^{-2} - e^{-4}$$

(c) [3 marks] Are the following statements regarding normal distributions TRUE or FALSE?

(i) They are always symmetric.

T

(ii) They always have mean of 0.

F

(iii) Samples from normal distribution rarely contain outliers.

T

Question 7. CONFIDENCE INTERVALS [7 marks]

A high-speed internet provider is making a survey on the quality of their service by asking their customers whether or not their service had been interrupted one or more times in the past month. In a random sample of 100 customers, the number of responses "YES", X, is 40.

(a) [1 mark] Suppose that all customers experienced interruption for one or more times in the past month with the same probability p. What is the distribution of X?

binom$(100, p)$

(b) [1 mark] What is $\hat{p}$, the sample proportion of being experienced internet interruption for one or more times in the past month as a point estimate of p?

$$\hat{p} = \frac{40}{100}$$

(c) [2 marks] Show that $E(\hat{p}) = p$.

see other solutions

(d) [1 mark] Explain the significance of $E(\hat{p}) = p$ in terms of estimation.

unbiased

(e) [2 marks] Give the form of a 95% confidence interval for p based on the sample proportion $\hat{p}$.

$$0.4 \pm 1.96\sqrt{\frac{0.4 \times 0.6}{100}}$$

End of Questions